Hydrostatic Pressure

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Introduction

Pressure is a very important quantity to the dynamics of the oceans, and fluids in general. Pressure *gradients* give rise to net accelerations that cause water to move, often in surprising ways. There was some introduction to this qualitatively in the estuaries discussion. Here we solidify those concepts quantitatively because we need to understand pressure gradients to make progress on understanding how the ocean moves.

To motivate ourselves, recall the demo of sloshing water in a tank (figure 1). The surface interface is tilted such that area A is elevated, and area B is depressed. Our intuition might say that the water needs to run downhill, and we may expect a flow, confined to the water surface, of water from area A to area B. Under this hypothesis, what do we expect the dye streak to do? What does the dye stream actually do (approximately)?

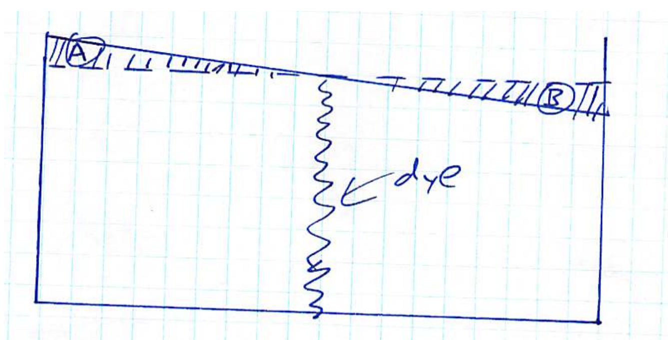

Figure 1: A tank with the water surface tilted from equilibrium. The surface wants to flatten, so water must be moved from area A to area B. What is the action of a dye streak through the water column?

To understand what is happening, we must understand pressure in the fluid. Pressure is simply the force per unit area that a parcel of fluid exerts on its neighbours. This is a bit hard to conceptualize, but consider a ballon filled with air. The air inside the balloon is pushing out against the ballon with a certain force per area. This happens because the molecules of air are moving in random motion due to their heat, and some of that momentum is exerted against the balloon's surface.

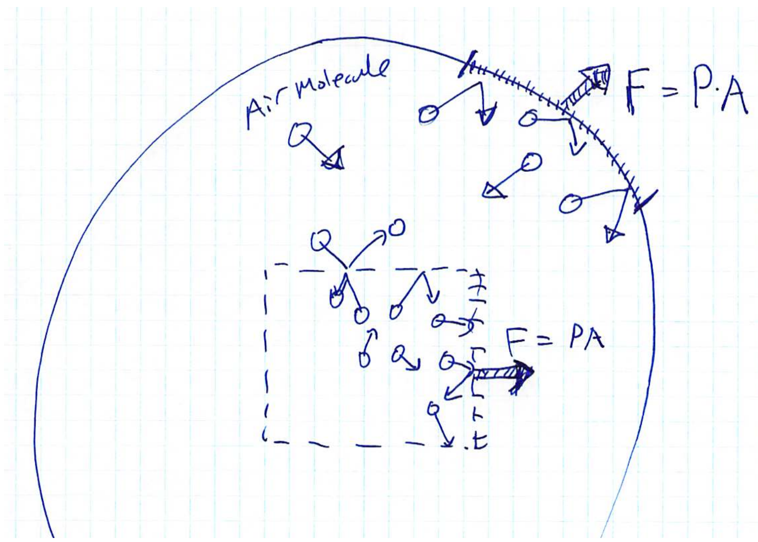

Figure 2: How to calculate the pressure for a one-layer fluid.

What is somewhat harder to conceptualize, is that the pressure forces are present within the balloon as well. If we consider an arbitrary volume inside the balloon, the molecules inside that volume hit molecules outside the volume, and extert a pressure force on them as well. In math terms, we say that the force F exerted across any arbitrary area inside or on the edge of the fluid is $F = PA$.

Hydrostatic Pressure

"Dynamic" pressure happens for fast flows, like those over an airplane wing. Fortunately, in the ocean things are usually slow enough that we can ignore the "dynamic" part, and focus on the "hydrostatic" part.

Hydrostatic pressure is relatively simple to understand. Consider a bucket of water, and think about the forces on a little "cube" of water inside the bucket. Suppose our cube is 0.1 m deep in the bucket. There is a column of water above our little cube that is not moving. The force of gravity wants to pull this column of water down. The force is simply $F_w = mg$, where $g = 9.8\,\mathrm{m\,s^{-2}}$, the mass $m = \rho V$, where $\rho = 1000\ \mathrm{kg\,m^{-3}}$ is the density of water, and V is the volume of the column of water $V = (0.1\ \mathrm{m})\,\mathrm{d}x\,\mathrm{d}y$. So, the water column is pushing down with force $F = \mathrm{d}x\,\mathrm{d}y\,(0.1\,\mathrm{m})\,(9.8\ \mathrm{m\,s^{-2}})\,(1000\,\mathrm{kg\,m^{-3}}) \approx \mathrm{d}x\,\mathrm{d}y\ 1000\,\mathrm{N\,m^{-2}}$.

What provides the force that holds this water up, against gravity? Its the water, just below, in our small cube. This water provides the pressure force from directly below

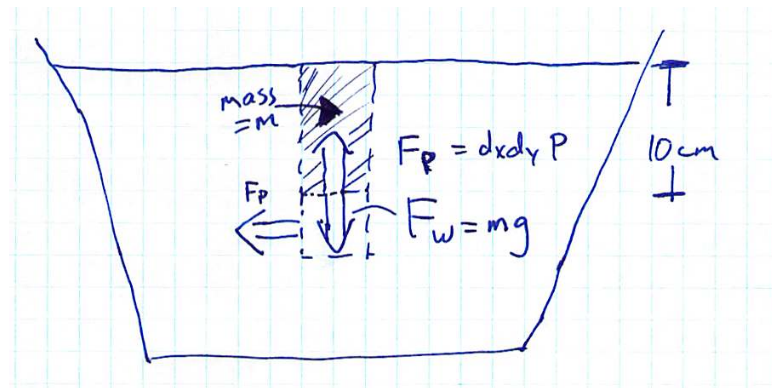

Figure 3: How to calculate the pressure for a one-layer fluid.

$F_p = P\,\mathrm{d}x\,\mathrm{d}y$. If the water is not moving, $F_p = -F_w$, so that the sum of forces in the vertical is zero. We can use this deduction to calculate the pressure at a depth h:

$$P(z=-h) = \rho g h. \tag{1}$$

For this example, $\rho = 1000\ \mathrm{kg\,m^{-3}}$, $g = 9.8\ \mathrm{m^2 s^{-1}}$, $h = 0.1\,\mathrm{m}$, so $P = 980\ \mathrm{N\,m^{-2}}$.

Essentially the pressure is simply the weight of the water above, per unit area.

Note that the squishing from the top, also causes squishing to the sides, so the water pressurized by the water above also exerts forces on the water in the horizontal direction.

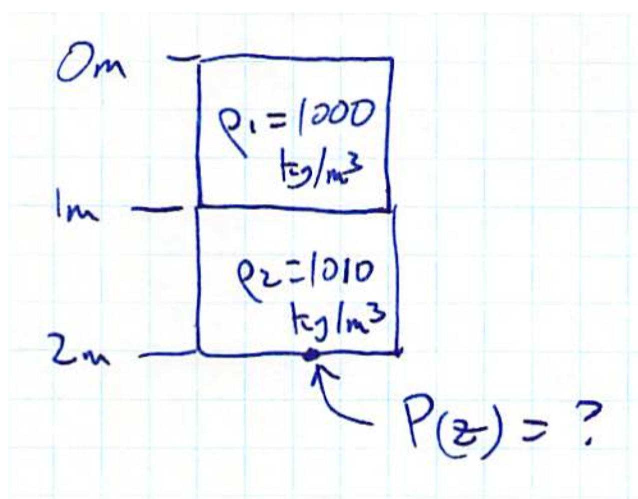

Figure 4: How to calculate the pressure for a two-layer fluid.

If the density of the water varies due to changes in temperature or salinity, then the weight needs to be calculated by summing up the different densities. i.e. imagine that the first meter, the density is 1000 $\mathrm{kg\,m^{-3}}$, and for the second meter, the density is 1010 $\mathrm{kg\,m^{-3}}$. Then the pressure at 2 m is:

$$\begin{aligned} P(z) &= \int \rho(z) g \mathrm{d}z && (2) \\ &\approx \sum_{j=1}^{n} \rho_j\, g\, \Delta z_j && (3) \\ &= (1000 * 1 * 9.8 + 1010 * 1 * 9.8)\ \ \mathrm{N\,m^{-2}} && (4) \\ &= 19698\ \mathrm{N\,m^{-2}} && (5) \end{aligned}$$

Adding more layers to the water column, or making the layers different thicknesses is just handled by adding more terms to the sum. Often it is useful to use a computer for this kind of calculation!

Pressure Gradients: Surface

We wouldn't care too much about pressure if it did not cause water to move. Consider a sloshing bathtub, mid-slosh (figure 5). In this situation it is intuitively obvious that the water wants to move from left to right, but what force is pushing it? First, where is pressure the highest at point 1 or point 2? These points are both a depth h below the *resting* depth of the top of the fluid. Above point 1, there is slightly more water due to the "slosh", whereas above point 2 there is slightly less. Therefore the pressure at point 1 is greater than at point 2. ($P_1(z=-h) = \rho g(h+\eta_1)$ and $P_2(z=-h) = \rho g(h+\eta_2)$, where η is the height of the water above its resting depth. $\eta_1 > 0$, and $\eta_2 < 0$, therefore $P_1 > P_2$).

It should also be obvious that *everywhere* in the fluid the pressure is decreasing to the right at a given depth i.e. $dP/dx < 0$. In fact the *horizontal pressure gradient* can be calculated from

$$P(z=-h) = \rho g(h+\eta) \tag{6}$$

and taking the derivative to get the gradient:

$$\frac{\mathrm{d}P}{\mathrm{d}x}(z=-h) = \rho g \frac{\mathrm{d}\eta}{\mathrm{d}x}. \tag{7}$$

or, the change in the pressure is caused by the slope of the sea surface.

How does this move the water? Lets consider the force diagram on a block of water inside the tub (figure 6) If the pressure is greater on the left hand side, then the force into the block from the left is larger than the force into the block on the right, and the net force $F_1 - F_2 > 0$, and the block tends to move to the right. Note that in this case $\mathrm{d}P/\mathrm{d}x < 0$. In the absence of other forces, the *acceleration* on the block in the horizontal direction is therefore given by

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x} \tag{8}$$

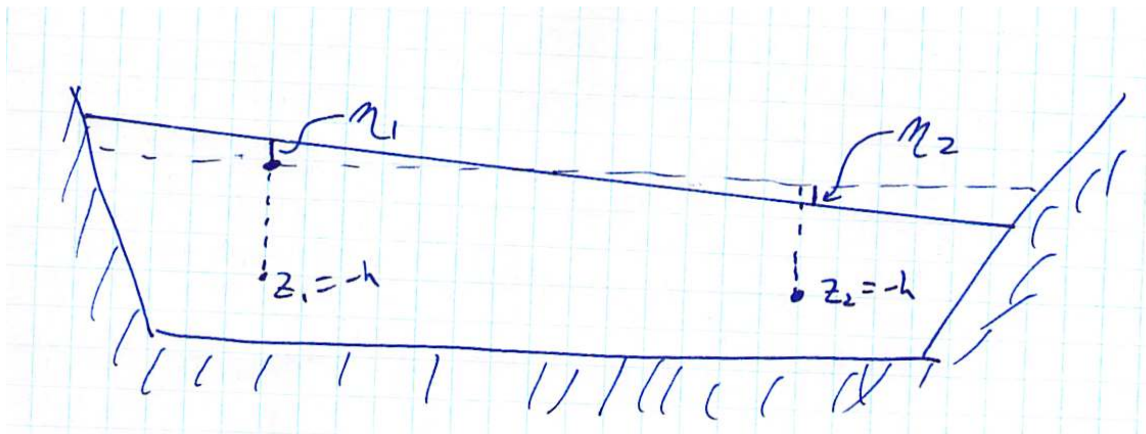

Figure 5: Pressure in a sloshing tub.

A question for the reader is to check that this gives the correct sign of the acceleration compared to their intuition.

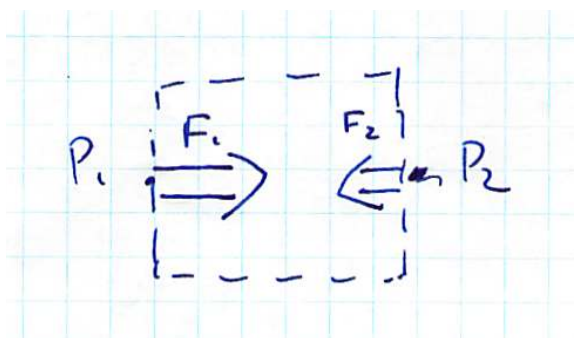

Figure 6: Lateral forces across a small hypothetical block.

For the example of the bathtub given above, it should also hopefully be clear that the pressure gradient, and thus the acceleration, does not depend on the depth below the surface. The pressure gradient, dP/dx, only depends on the sea surface tilt, not on z.

Pressure Gradients: Internal

If the fluid has density layers, then the pressure field can be more complicated, and the motions harder to predict. For the simplest case, consider a two-layer fluid, with a flat upper surface, and a tilted interface between two layers with densities ρ_1 and ρ_2

(figure 7) with $\rho_2 > \rho_1$. Again, the pressure is lower on the right than left, so water in the deeper layer has a pressure gradient force from high pressure to low pressure, i.e. to the right.

Mathematically, $P_A(z=-h) = g(\rho_1 h_1(x_A) + \rho_2(h - h_1(x_A)))$, and $P_B(z=-h) = g(\rho_1 h_1(x_B) + \rho_2(h - h_1(x_B)))$. We can estimate the pressure gradient as

$$\frac{\mathrm{d}P}{\mathrm{d}x} \approx \frac{P_B - P_A}{x_B - x_A} = g(\rho_1 - \rho_2)\frac{h_1(x_B) - h_2(x_A)}{x_B - x_A} = -g(\rho_2 - \rho_1)\frac{\mathrm{d}h}{\mathrm{d}x} \tag{9}$$

So again, the pressure gradient depends on the slope of the interface, except this time, the interface is between the two densities.

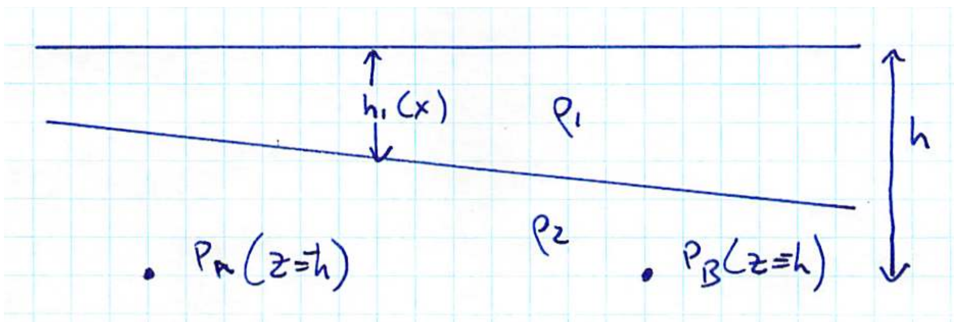

Figure 7: How to calculate the pressure for a two-layer fluid.

Of course, the cases can be combined (figure 8). There can be a surface tilt, and many interface tilts. While care needs to be taken to keep track of all the layer thicknesses, and the algebra looks messy on the page, the fundamental idea is to simply calculate the weight of the water about the point of interest, including the effect of the local sea-surface elevation.

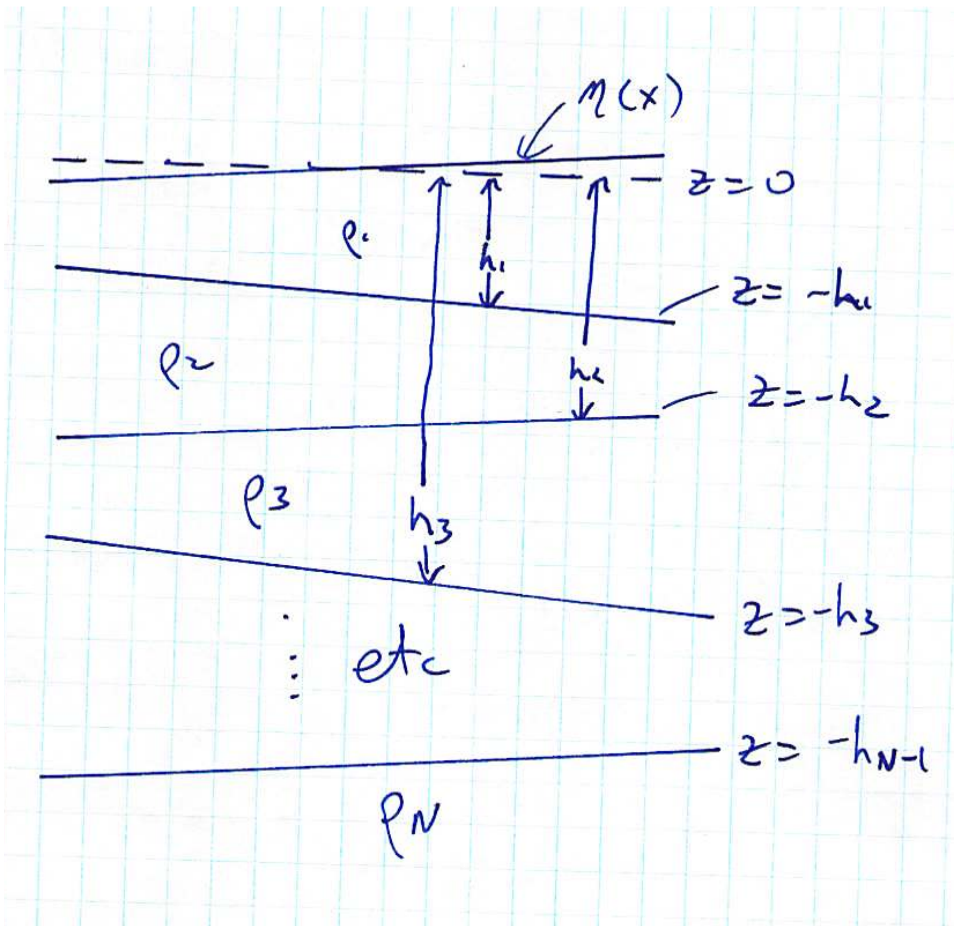

Figure 8: Many layer-fluid.