Quantifying Mixing

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Introduction

Mixing is another way to flux tracers from one location to another in a fluid. If you pour fresh water over top of salty water, they will eventually mix. This can happen without there being any flow in the fluid, just due to the molecular motion of the salt particles slowly "diffusing".

To start thinking about the problem, imagine a container with salt-stratified fluid in it (figure 1). High salt at the bottom, low salt at the top. Salt will tend to move upwards with time, until eventually the whole container has the same salinity. This upward movement of salt is the "diffusive flux of salinity".

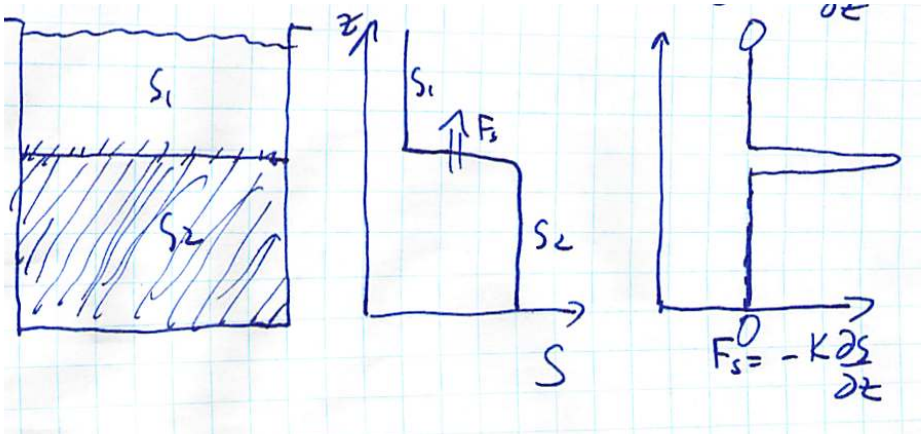

Figure 1: Flux across an abrupt interface between two layers of different salinity waters. The diffusive flux of salt is at the interface and upwards. Left) Schematic sketch of a beaker; Middle) vertical profile of salt versus depth; Right) vertical profile of salt flux versus depth.

The strength of the gradient affects the strength of the flux. So, where there are steeper salt gradients, the salt flux is stronger. So in figure 1 the only gradient is at the interface. In figure 2, the gradient is continuous, but in general greater towards the bottom than the top, so the diffusive fluxes are stronger near the bottom.

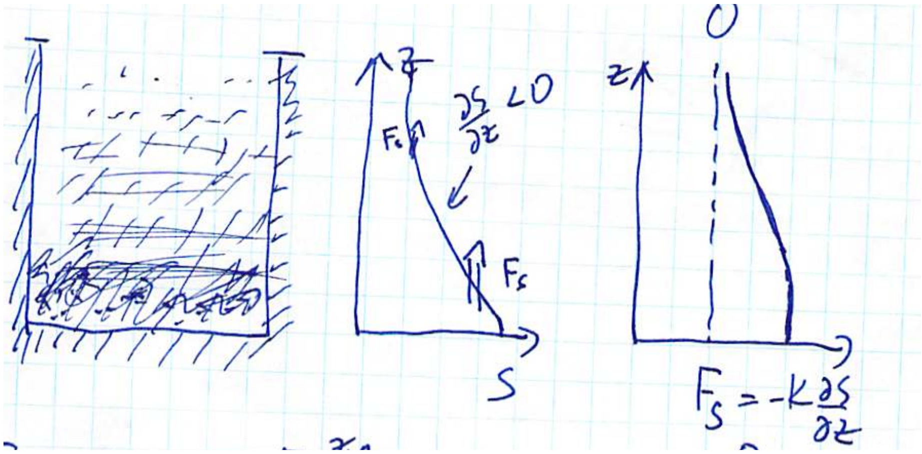

Figure 2: Left) Schematic sketch of a beaker; Middle) vertical profile of salt versus depth; Right) vertical profile of salt flux versus depth.

We quantify the "flux" of tracer with concentration $C\ \mathrm{g\,m^{-3}}$ as

$$F_C = -\kappa \frac{\mathrm{d}C}{\mathrm{d}x} \tag{1}$$

where κ is the *diffusion co-efficient*, and has units of $\mathrm{m^2\,s^{-1}}$. Note that the flux has units of

Just like for an advective flux, the transport is just the flux times the area perpendicular to the direction of transport. So if we consider a pipe with area A_x, then the diffusive transport is

$$q_C = \mathbf{F}_C A_x = -\kappa \frac{\mathrm{d}C}{\mathrm{d}x} A_x \tag{2}$$

where x is the direction along the pipe. This transport is in $g\,s^{-1}$, just like the advective transport due to a current.

Once we have the diffusive transport, we use it exactly the same way as an advective transport, and sometimes at the same time.

Example

Question Consider a container with salt stratification in the vertical. The container is 10 m tall, and has a cross-sectional area of $A = 1\ \mathrm{m}^2$. If the salinity is $S = 32$ psu at $z = 4$ m, and $S = 30$ psu at $z = 6$ m. Suppose we measure the salinity in the lower half, and find that it is losing salt at a rate of $0.001\ \mathrm{psu\,s^{-1}}$, what is the approximate value of the diffusivity acting at $z = 5$ m?

Answer The transport of salt through the interface is $q_S = 0.001\ \mathrm{psu\,s^{-1}}$ and we know that $q_s = -A\kappa\frac{\mathrm{d}S}{\mathrm{d}z}$. We estimate the gradient as $-1\ \mathrm{psu\,m^{-1}}$, where the negative sign indicates the salt is decreasing in the vertical direction. So, $\kappa \approx 0.001\mathrm{m}^2\,\mathrm{s}^{-1}$, in the vertical.

Box model representation

In the Chemical literature (i.e. Eos 312), gradients are often not used to specify the diffusive flux. The chemists set up problems as "two-box" problems. For the case above, the upper box would be one salinity, $S1$, and the box below a second $S2$. The flux is then parameterized as $F_s = v_{mix}\,(S2 - S1)$, where v_{mix} is a "mixing velocity", with units $m\,s^{-1}$, or a "mixing transport" with units $m^3\,s^{-1}$.

Advection and Diffusion

Of course both types of fluxes can co-exist; the salinity at a location can change because new water moves in, or because of mixing. To account for this, we simply add the fluxes through any interface together, taking care to use the right sign. A typical problem might be:

Example Question

Question Suppose we have a hose filling a tank with cold water, temperature T_h (figure 1). The hose has cross-sectional area A_h, and flow speed $u_h > 0$, where the sign indicates flow of water into the tank. Suppose the tank is heated from above, and the water pumped in from below is allowed to spill over the top of the tank. If the tank has a cross sectional area A_T, and at two depths in the tank the temperature is measured, such that at z_1 it is T_1 and at z_2 it is T_2, derive an expression to estimate the the diffusivity of the water in the tank if the temperature in the tank is in steady state.

Answer First, water must move up the tank by conservation of volume: $w_t A_t = u_h A_h = q_h$, where q_h is a constant volume flux in the hose.

Secondly, the transports of heat in the hose, must equal the sum of the advective plus diffusive fluxes at z_1: $0 = Q_h + Q_t + Q_d$. Here $Q_h = T_h\,q_h$ is the transport of temperature through the hose (really we should use "heat" rather than temperature, but "heat" is just $J = (T + 273)\,\rho\,c_p$, where ρ is the density of water, and c_p is the heat capacity. The constants cancel in the algebra). Similarly $Q_t = -T_1\,q_h$ is the advective flux out the top of the interface defined by T_1. This is negative because the flux of heat is out of the top. Finally, $Q_d = \kappa \frac{T_2 - T_1}{z_2 - z_1} A_t$, which is positive because the diffusive flux of heat is down from the surface to the bottom.

After a bit of algebra, we find that

$$\kappa = \frac{q\,(T_1 - T_t)}{(T_2 - T_1)/(z_2 - z_1)A_t} \tag{3}$$

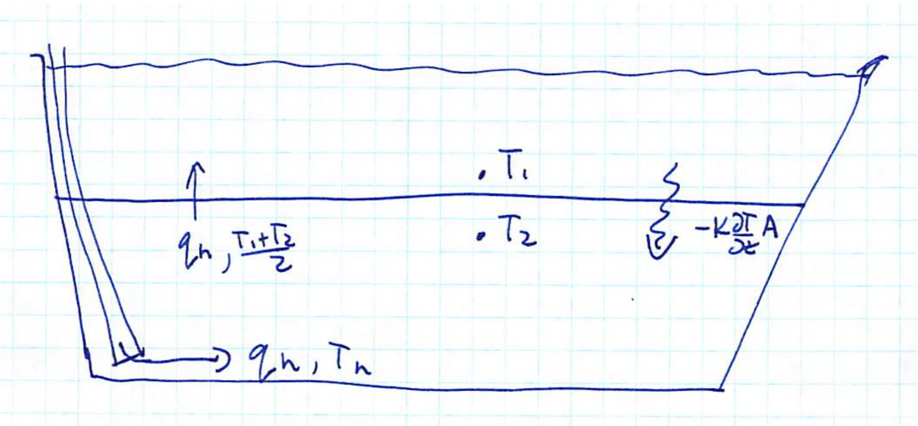

Figure 3: Advection and diffusion leading to steady state in a basin. In steady state, the heat flux is advected in at the bottom at a cold temperature. That water flows through the middle of the box at the same rate, but at a warmer temperature. It must warm up due to diffusion with the water above.

Molecular and Turbulent Diffusions

Scaling the diffusivity

Suppose in the above question, the gradient is proportional to $T_1 - T_t$, we could rewrite the equation as

$$\kappa \sim w\Delta z \tag{4}$$

This says that for a known flow w the scales are larger for larger κ. The "$\sim$" symbol means "scales as", and is much looser than an "equals" sign. The ocean is often thought of as having a vertical flux on the order of 3 $\mathrm{m\,y^{-1}} = 10^{-7}\mathrm{m\,s^{-1}}$. If the diffusivity of heat was molecular, then $\kappa_T = 10^{-7}\ \mathrm{m^2\,s^{-1}}$, and the warm water of the upper ocean would be confined to $\Delta z \approx 1$ m. Molecular diffusivity is *very* slow.

Another way to think about this is to consider the time scale of mixing. Suppose we have a beaker that is H tall, then diffusivity scales as

$$\kappa \sim \frac{H^2}{T} \tag{5}$$

so a timescale of mixing is given by $T \sim H^2/\kappa$. Considering a beaker, $H = 0.2$ m, and molecular diffusivity $\kappa = 10^{-9}\ \mathrm{m^2\,s^{-1}}$, we would expect it to take on the order of a year to homogenize. Whether that actually occurs or not would require a beaker that was very isolated from outside influences that might cause turbulent mixing.

Turbulent Mixing

The ocean is turbulent, and the resultant stirring greatly enhances the mixing over what we would find if we relied on molecular mixing. with the turbulence caused by winds,

Quantity — Diffusivity [ms^{-2}]
Temperature/heat — 10^{-7}
Salinity, other contaminants — 10^{-9}
momentum — 10^{-6}

tides, and currents. An example (figure 4) shows flow in a channel in Puget Sound, where dense water flows quickly under light water. The "billows" that result have very high turbulence, with a *turbulent diffusivity* of $K \approx 10^{-1}\ \mathrm{m}^2\,\mathrm{s}^{-1}$.

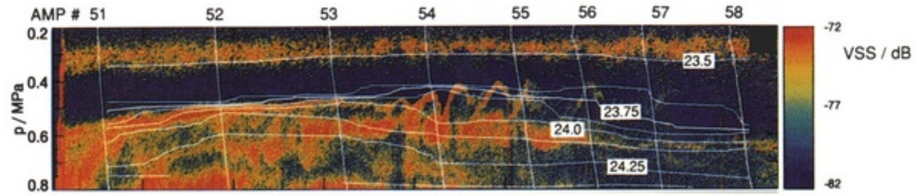

Figure 4: A turbulent billow in Admiralty Inlet, Puget Sound. The bottom flow (deeper than 0.6 MPa) is flowing to the right, and the shallower flow to the left. The "shear" produces instabilities, and drives enhanced mixing starting between 53 and 54. These regions ahve very enhanced turbulence.

What happens in detail is shown in a numerical simulation figure 5, where the evolution of the billows is shown in a series of time steps. After the turbulence generated by the billows dies down, the water is much more mixed than it was before the billows were triggered.

Unlike molecular diffusion, the strength of turbulent mixing (i.e. K) does not depend strongly on the property being mixed. The mixing due to the billows pictured here is vigorous, and mixes temperature, salinity, contaminants, and momentum in a similar manner. However, the strength of the mixing depends on the size of the billows.

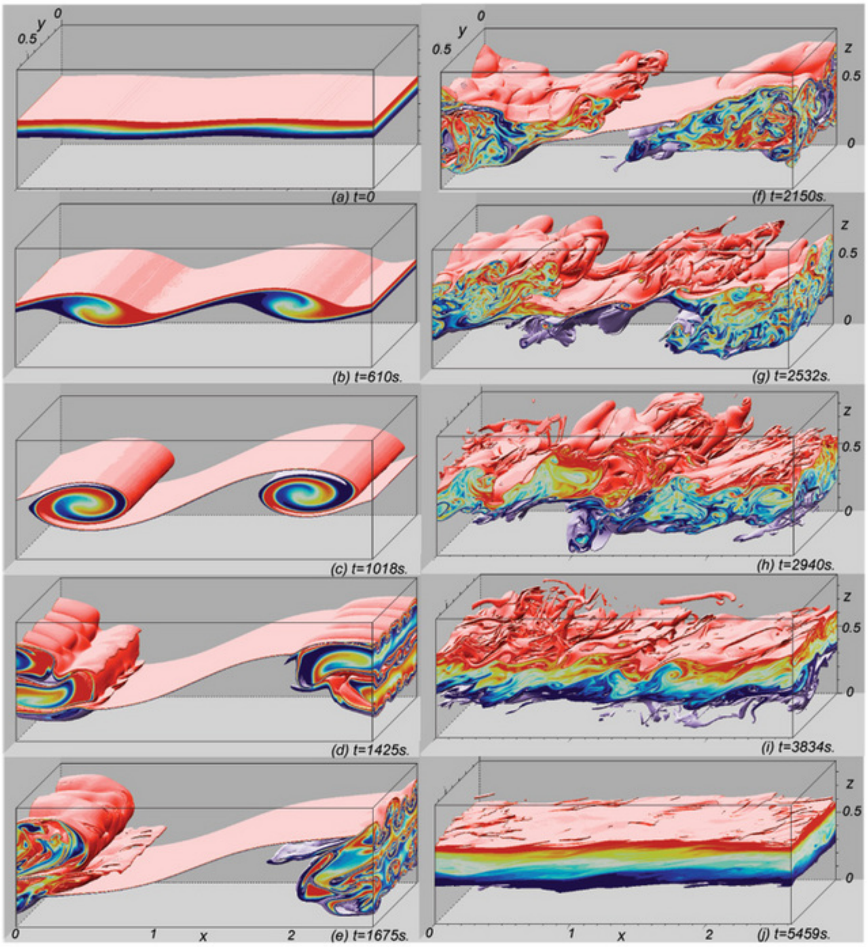

Figure 5: A numerical model of a breaking wave similar to that shown in figure 4.