Take-home Final Assignment
Phy 426, 2024
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Exam to be completed independently, though questions to the instructor are welcome/encouraged. Open-book, open-notes are fine. Show all work, define any constants you need that I don't provide, check your units, etc. Except as noted, the density of the fluid is ρ, gravity is g, the kinematic viscosity ν, and the fluid can be assumed Bousinesque and incompressible.

Please try to make it readable. I will deduct up to 10% for illegible chicken scratches, so please take the time to recopy your work. Typeset is preferred but not necessary.

Question 1. Shear production of turbulent kinetic energy

Consider a slowly evolving flow in the x-direction with a slowly evolving shear in the z-direction given by $\partial U/\partial z \neq 0$. On top of this slowly evolving flow is a flow that is turbulent with velocity fluctuations u', v', w'

1. [10] Using Reynolds averaging form an energy equation for the mean flow and show that the mean flow has a sink of energy given by $+\overline{u'w'}\frac{\partial U}{\partial z}$
2. [4] Explain why this is a sink instead of a source by considering the likely sign of the terms.
3. [8] Again using Reynolds averaging on the energy equation, show that the *turbulent* kinetic energy has a *source* due to this term.

Question 2. Minimum energy of irrotational flow

Mathematically (as opposed to dynamically), there are infinitely many flow fields $\mathbf{u}(\mathbf{x}, t)$ that can satisfy i) $\nabla \cdot \mathbf{u} = 0$ and ii) $\mathbf{u} \cdot \mathbf{n} = 0$ along any boundary S enclosing a fluid.

1. [15] Show that the unique *irrotational* flow has the minimum kinetic energy of any such flow. (Hint, any flow can be split into an irrotational component, and a residual).

Question 3. Flow over a corrugated seafloor

Suppose we have a steady mean flow, U, over a corrugated seafloor described by

$$z = -H + \epsilon \cos kx \tag{1}$$

where $\epsilon << H$. The seasurface would be at $z = 0$ if there were no flow. Set the problem up using a flow potential:

$$u = U + \frac{\partial \phi}{\partial x} \tag{2}$$

$$w = \frac{\partial \phi}{\partial z} \tag{3}$$

1. [10] Argue that the linearized surface boundary conditions (at $z = 0$) on ϕ are given by $U\frac{\partial\phi}{\partial x} = -g\eta$, and $U\frac{\partial\eta}{\partial x} = \frac{\partial\phi}{\partial z}$, and that the bottom boundary condition (at $z = -H$) is given by $\frac{\partial\phi}{\partial z} = -Uk\epsilon\sin kx$.
2. [20] Solve Laplace's equation for the seasurface response $\eta(x)$ and argue that if $kU^2 < g\tanh kH$ the crests in the surface waves fall over the troughs of the bottom corrugations (and they are in phase if $kU^2 > g\tanh kH$)

Question 4. Oscillating forcing of a fluid above a plate

Consider a fluid with viscosity ν that has an infinite plate as a no-slip bottom boundary condition. The fluid is infinite in y above the plate. Suppose that the fluid is everywhere subject to an oscillating horizontal pressure gradient (parallel to the plate), given by

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x} = A\cos\omega t \tag{4}$$

where ω is the forcing frequency in rad/s.

1. [15] Show that a solution for the flow as a function of y and t, assuming the flow is well-developed (e.g. long after any startup transient) is given by:

$$u(y,t) = -\frac{A}{\omega}\left(\sin\omega t\left(1 - e^{-ky}\cos ky\right) + \cos\omega t\; e^{-ky}\sin ky\right) \tag{5}$$

   where $k \equiv \sqrt{\frac{\omega}{2\nu}}$.
2. [5] Comment on the character of the solution, particularly with respects to the phase of the solution both far from and near the plate. A sketch is helpful - and while you are not required to, I found doing this quickly in a plotting package helpful.
3. [10] Calculate the work per length x done by the pressure force on the flow, and show that it is equal to the viscous dissipation in the flow, averaged over a forcing cycle.

Answer Key for Exam A

Question 1. Shear production of turbulent kinetic energy

Consider a slowly evolving flow in the x-direction with a slowly evolving shear in the z-direction given by $\partial U/\partial z \neq 0$. On top of this slowly evolving flow is a flow that is turbulent with velocity fluctuations u', v', w'

1. [10] Using Reynolds averaging form an energy equation for the mean flow and show that the mean flow has a sink of energy given by $+\overline{u'w'}\frac{\partial U}{\partial z}$

   Answer: This is covered explicitly in the text in the chapter on turbulence: "Kinetic Energy Budget of Mean Flow".

2. [4] Explain why this is a sink instead of a source by considering the likely sign of the terms.

   Answer: This is again covered explicitly in the text: if $dU/dz > 0$ then $w' > 0$ will, on average bring water upwards so that it is moving slower then U, and hence $u' < 0$. Similarly if the converse is true, and hence $\overline{u'w'} < 0$, and $\overline{u'w'}\frac{\partial U}{\partial z} < 0$. The converse all applies if $dU/dz < 0$.

3. [8] Again using Reynolds averaging on the energy equation, show that the *turbulent* kinetic energy has a *source* due to this term.

   Answer: This is covered in the *next* section, entitled "Kinetic Energy Budget of Turbulent Flow" where after all is done, the term $-\overline{u'w'}\frac{\partial U}{\partial z}$ appears, and is hence usually a source.

Question 2. Minimum energy of irrotational flow

Mathematically (as opposed to dynamically), there are infinitely many flow fields $\mathbf{u}(\mathbf{x}, t)$ that can satisfy i) $\nabla \cdot \mathbf{u} = 0$ and ii) $\mathbf{u} \cdot \mathbf{n} = 0$ along any boundary S enclosing a fluid.

1. [15] Show that the unique *irrotational* flow has the minimum kinetic energy of any such flow. (Hint, any flow can be split into an irrotational component, and a residual).

   Answer: An irrotational flow can be represented by $\mathbf{u}_I = \nabla\phi$ where ϕ is the flow potential, so any arbitrary flow field can be written as the unique irrotational flow plus an anomaly: $\mathbf{u} = \nabla\phi + \mathbf{u}'$.

   The kinetic energy, T, of such a flow is $2T = \nabla\phi^2 + 2\mathbf{u}' \cdot \nabla\phi + \mathbf{u}'^2$. Integrating over the volume enclosing the fluid we get

$$\int_V T \, \mathrm{d}V = \int_V \nabla\phi^2 \, \mathrm{d}V + \int_V \mathbf{u}'^2 \, \mathrm{d}V + 2\int_V \mathbf{u}' \cdot \nabla\phi \, \mathrm{d}V.$$

(1)

The first term is the kinetic energy of the irrotational flow, the second term is always positive, so only the last term could counterbalance to to make the total kinetic energy smaller than $\int_V \nabla\phi^2 \, \mathrm{d}V$. However, we can easily show that this term is zero because the fact that $\nabla \cdot \mathbf{u}' = 0$ means that $\mathbf{u}' \cdot \nabla\phi = \nabla \cdot (\mathbf{u}'\phi)$ allowing us to use the divergence theorem:

$$\begin{aligned}\int_V \mathbf{u}' \cdot \nabla\phi \, \mathrm{d}V &= \int_V \nabla \cdot (\mathbf{u}'\phi) \, \mathrm{d}V \\ &= \oint_S \phi\mathbf{u}' \cdot \mathrm{d}\mathbf{S} \\ &= 0\end{aligned}$$

So,

$$\begin{aligned}\int_V T \, \mathrm{d}V &= \int_V \nabla\phi^2 \, \mathrm{d}V + \int_V \mathbf{u}'^2 \, \mathrm{d}V \\ &\geq \int_V \nabla\phi^2 \, \mathrm{d}V\end{aligned}$$

and therefore the irrotational flow has the minimum energy.

Question 3. Flow over a corrugated seafloor

Suppose we have a steady mean flow, U, over a corrugated seafloor described by

$$z = -H + \epsilon \cos kx \tag{2}$$

where $\epsilon << H$. The seasurface would be at $z = 0$ if there were no flow. Set the problem up using a flow potential:

$$\begin{aligned} u &= U + \frac{\partial\phi}{\partial x} &(3)\\ w &= \frac{\partial\phi}{\partial z} &(4)\end{aligned}$$

1. [10] Argue that the linearized surface boundary conditions (at $z = 0$) on ϕ are given by $U\frac{\partial\phi}{\partial x} = -g\eta$, and $U\frac{\partial\eta}{\partial x} = \frac{\partial\phi}{\partial z}$, and that the bottom boundary condition (at $z = -H$) is given by $\frac{\partial\phi}{\partial z} = -Uk\epsilon \sin kx$.

Answer: These are in analogy to surface waves. The x-momentum equation gives the first surface boundary condition:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \tag{5}$$

The flow is steady, and the linear terms in the advective terms give $U\partial u/\partial x$, and the surface pressure gradient can be approximated by the sea surface gradient.

The second surface boundary condition derives from

$$u\frac{\partial\eta}{\partial x} = w \tag{6}$$

The bottom boundary condition comes from $w = -U\frac{\mathrm{d}H}{\mathrm{d}x}$

2. [20] Solve Laplace's equation for the seasurface response $\eta(x)$ and argue that if $kU^2 < g\tanh kH$ the crests in the surface waves fall over the troughs of the bottom corrugations (and they are in phase if $kU^2 > g\tanh kH$)

   Answer: Solutions are of the form

$$\phi = \left(Ae^{kz} + Be^{-kz}\right)\sin kx \tag{7}$$

We can combine the surface boundary conditions, eliminating η to get:

$$-\frac{U^2}{g}\frac{\partial^2\phi}{x^2} = \frac{\partial\phi}{\partial z} \tag{8}$$

which at $z = 0$ gives

$$A = -B\frac{U^2k + g}{U^2k - g} \tag{9}$$

The bottom boundary condition at $z = -H$ gives:

$$Ae^{-kH} - Be^{kH} = U\epsilon \tag{10}$$

Solving for B and A gives

$$B = \frac{-U\epsilon\left(U^2k - g\right)}{2\left(U^2k\cosh kH - g\sinh kh\right)} \tag{11}$$

$$A = \frac{U\epsilon\left(U^2k + g\right)}{2\left(U^2k\cosh kH - g\sinh kh\right)} \tag{12}$$

So now we know ϕ, we can solve for η using the surface boundary condition again:

$$\eta = \frac{\epsilon\cos kx}{\cosh kH - \frac{g}{U^2k}\sinh kH} \tag{13}$$

We see from this that the waves are in phase with the bottom corrugations if

$$\tanh kH < \frac{U^2k}{g} \tag{14}$$

Question 4. Oscillating forcing of a fluid above a plate

Consider a fluid with viscosity ν that has an infinite plate as a no-slip bottom boundary condition. The fluid is infinite in y above the plate. Suppose that the fluid is everywhere subject to an oscillating horizontal pressure gradient (parallel to the plate), given by

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x} = A\cos\omega t \tag{15}$$

where ω is the forcing frequency in rad/s.

1. [15] Show that a solution for the flow as a function of y and t, assuming the flow is well-developed (e.g. long after any startup transient) is given by:

$$u(y,t) = -\frac{A}{\omega}\left(\sin\omega t\left(1 - e^{-ky}\cos ky\right) + \cos\omega t\ e^{-ky}\sin ky\right) \tag{16}$$

where $k \equiv \sqrt{\frac{\omega}{2\nu}}$.

Answer: The flow is subject to the equation

$$\frac{\partial u_0}{\partial t} = -A\cos\omega t + \nu\frac{\partial^2 u_0}{\partial y^2} \tag{17}$$

We can write as a complex differential equation, with $u_0 = \mathcal{R}(u)$:

$$\frac{\partial u}{\partial t} = -Ae^{i\omega t} + \nu\frac{\partial^2 u}{\partial y^2} \tag{18}$$

where $u(0,t) = 0$ and $\frac{\partial u}{\partial y} \to 0$ as $y \to \infty$ The solution will oscillate at the same frequency, though not necessarily in phase, and it is a forced oscilator, so there is a component to the response in balance with the forcing: $u(y,t) = \left(U(y) + \frac{A\,i}{\omega}\right)e^{i\omega t}$, leaving a differential equation

$$i\omega U = \nu\frac{\partial^2 U}{\partial y^2} \tag{19}$$

This has combined oscilatory/decaying solutions in y

$$U(y) = be^{\pm ky}e^{\pm iky} \tag{20}$$

where $k = \sqrt{\frac{\omega}{2\nu}}$. The boundary conditions on $U(y)$ preclude the exponentially growing solution, so the result is

$$U(y) = be^{-ky}e^{-iky} \tag{21}$$

The boundary condition at $y = 0$ gives

$$u(y,t) = \frac{A\,i}{\omega}\left(1 - e^{-ky}e^{-iky}\right)e^{i\omega t} \tag{22}$$

which gives, for the real part:

$$u_0(y,t) = -\frac{A}{\omega}\left(\sin\omega t\left(1 - e^{-ky}\cos ky\right) + \cos\omega t\ e^{-ky}\sin ky\right) \tag{23}$$

2. [5] Comment on the character of the solution, particularly with respects to the phase of the solution both far from and near the plate. A sketch is helpful - and while you are not required to, I found doing this quickly in a plotting package helpful.

Answer: Far from the plate, the velocity is exactly 90 degrees out of phase with the pressure forcing, as expected.

Close to the plate, the velocity is of course attenuated. However, it also *leads* the velocity far from the plate. This is perhaps unsurprising as the pressure gradient is the same as aloft, but the flow speed smaller so the flow reverses direction sooner.

3. [10] Calculate the work per length x done by the pressure force on the flow, and show that it is equal to the viscous dissipation in the flow, averaged over a forcing cycle.

Answer: The pressure work done, averaged over the forcing cycle, is

$$W = \frac{1}{T}\int_0^T \mathrm{d}t \int_0^\infty -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x}u \; \mathrm{d}y \tag{24}$$

where $T = \frac{2\pi}{\omega}$.

$$W = \frac{1}{T}\int_0^T \mathrm{d}t \int_0^\infty \frac{A^2}{\omega}\left(\cos\omega t \sin\omega t \left(1 - e^{-ky}\cos ky\right) + \cos^2(\omega t)\; e^{-ky}\sin ky\right)\mathrm{d}y \tag{25}$$

The first term integrates in time to zero, and the second to:

$$W = \frac{A^2}{2\omega}\int_0^\infty e^{-ky}\sin ky \; \mathrm{d}y \tag{26}$$

or

$$W = \frac{A^2}{4\omega k} \tag{27}$$

For the dissipation

$$D = \frac{1}{T}\int_0^T \mathrm{d}t \int_0^\infty \nu\left(\frac{\partial u}{\partial y}\right)^2 \; \mathrm{d}y \tag{28}$$

This is much easier to solve as a complex velocity and then take the real part $D = \mathcal{R}D'$:

$$D' = \frac{1}{T}\int_0^T \mathrm{d}t \int_0^\infty \nu\left(\frac{Aki}{\omega}(1+i)\, e^{-k(1+i)}e^{i\omega t}\right)^2 \; \mathrm{d}y \tag{29}$$

$$D' = \frac{1}{T}\int_0^T \mathrm{d}t \int_0^\infty \nu 2i\frac{A^2k^2}{\omega^2}e^{i2\omega t}e^{-2ky(1+i)} \; \mathrm{d}y \tag{30}$$

These are both easy integrals, and the real part yields:

$$D = \nu\frac{kA^2}{2\omega^2} \tag{31}$$

and recalling that $\nu = \frac{\omega}{2k^2}$

$$D = \frac{A^2}{4k\omega} \tag{32}$$