Waves in the Ocean: Linear Treatment

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Contents

Foreward

These lecture notes are designed for the *Linear Ocean Waves* course taught at Scripps Institution of Oceanography, UCSD as part of the core curriculum for first year physical oceanography students. This course is designed to give a survey of linear ocean waves including surface gravity waves, acoustic waves, internal waves, and rotating shallow water waves with a strong fluid dynamics treatment over the course of about 20 lectures (one quarter). These lecture notes are broken down into two components. The first is wave systems in *homogeneous* media (Chapters 2-12). The second component focusses on wave propagation in *inhomogenous* media and thus brings in ray theory, WKB, action conservation, etc (Chapters 13–16).

These lecture notes are derived from when I took this course in Winter quarter of 1994 taught by Rick Salmon. Prof. Salmon distributed photocopies of hand-written lecture notes and his course was excellent, very thought provoking, and helped us understand the deep links between many different types of wave systems, in particular in the relationship between dispersion relations and energy propagation. Many ocean wave texts target one particular type of ocean wave - say surface gravity waves (the most common). But Prof. Salmon's notes spoke broadly across classes of ocean wave systems. Those notes are the foundation of these lecture notes and I am deeply endebted and grateful to Prof. Salmon. In the intervening years the notes have been extended and deeply revised, and are still a work in progress. However, the core theme of the linkage between many different types of wave systems hopefully remains.

These lecture notes and my thinking on waves is also indebted to many other wonderful texts on ocean waves, some of which go much further in depth on certain specific topics. These include:

- Kundu, Cohen, Dowling, Fluid Mechanics
- Mei, CC, The Applied Dynamics of Surface Gravity Waves
- Lighthill, Waves in Fluids
- Whitham, Linear and Nonlinear waves
- Dean and Dalyrmple, Wave Mechanics for Engineers and Scientists
- Chapman and Malanotte-Rizzoli, Waves Lecture Notes in tribute to Myrl Hendershott
- Pedlosky, J. Waves in the Ocean and Atmosphere. Introduction to Wave Dynamics

I hope these notes are useful and do appreciate any feedback.
Falk Feddersen

Chapter 1

Introduction to the Wave Equation(s)

1.1 First Order Linear Wave Equation

First, we consider the first order linear wave equation which forms the backbone of conservation equations in fluid dynamics. Consider on an infinite spaital domain $(-\infty < x < \infty)$ and an infinite time t domain, the linear first order wave equation is,

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \tag{1.1}$$

for real, constant $c > 0$. As (1.1) is first order in time, we need a single initial condition of

$$\phi(x, t = 0) = \phi_0(x), \tag{1.2}$$

and because of the infinite domain we don't specify boundary conditions. The solution can be written as

$$\phi(x, t) = \phi_0(x - ct) \tag{1.3}$$

which implies that the initial condition simply propagates to the right with constant speed c with permanent form.

Why is this the case? We plug the solution (1.3) back into (1.1), but first: let $u = x - ct$ so that any function $f(x - ct) = f(u)$ and the derivative of the function with respect to u is notated as $\mathrm{d}f/\mathrm{d}u = f'$. Now

$$\begin{aligned}
\frac{\partial \phi}{\partial t} &= \frac{\partial \phi_0(x-ct)}{\partial t} = \frac{\partial \phi_o(u)}{\partial t} = \frac{\mathrm{d}\phi_0}{\mathrm{d}u}\frac{\partial u}{\partial t} = -c\frac{\mathrm{d}\phi_0}{\mathrm{d}u} \\
\frac{\partial \phi}{\partial x} &= \frac{\partial \phi_0(x-ct)}{\partial x} = \frac{\partial \phi_o(u)}{\partial x} = \frac{\mathrm{d}\phi_0}{\mathrm{d}u}\frac{\partial u}{\partial x} = \frac{\mathrm{d}\phi_0}{\mathrm{d}u}
\end{aligned}$$

Note the difference in the partial and total derivative above. Thus the first-order wave equation becomes

$$-c\frac{\mathrm{d}\phi_0}{\mathrm{d}u} + c\frac{\mathrm{d}\phi_0}{\mathrm{d}u} = 0$$

1.1.1 Plane wave solution and dispersion relationship

A common practice is to plug in a propagating wave solution such as $\cos(kx-\omega t)$ or $\sin(kx-\omega t)$ into the governing equations and hunting for a solution and dispersion equation. For linear systems it is often more convenient to use complex notation. For example, let ϕ have a wave like solutions where

$$\phi = e^{\mathrm{i}(kx-\omega t)} \tag{1.4}$$

where k is a **wavenumber** with units [rad/m] and a **wavelength** $\lambda = 2\pi/k$. Similarly the **radian frequency** ω has units [rad/s] and is associated with the **wave period** $T = 2\pi/\omega$. Plug the wave solution (1.4) into (1.1) gives us a dispersion relationship

$$[-\mathrm{i}\omega + \mathrm{i}kc]e^{\mathrm{i}(kx-\omega t)} = 0, \implies \omega = kc.$$

Note that this dispersion relationship has $\omega(k)$ as a linear function of k. Or that $\omega/k = c$ is a constant and does not depend upon k. The type of waves that propagate by such a dispersion relationship are called **non-dispersive** waves. This means that all frequencies or wavenumbers propagate at the same speed - in this case c.

1.1.2 Linearity and Fourier Solution

Because the first order wave equation is linear, if $a(x,t)$ and $b(x,t)$ are both solutions to (1.1) on an infinite domain, then any combination of $c_1 a(x,t) + c_2 b(x,t)$ is also a solution. We will now exploit this to perform Fourier analysis on the first order wave equation. This analyis will be fairly simple but introduce concepts that will be used throughout these lectures.

First, write $\phi(x,t)$ as a Fourier integral, that is

$$\phi(x,t) = \int_{-\infty}^{\infty} \hat{\phi}(k,t)e^{\mathrm{i}kx}\mathrm{d}k \tag{1.5}$$

where $\hat{\phi}$ is the Fourier transform of ϕ, the integral is over all k. The Fourier transform is defined as

$$\hat{\phi}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(x)e^{-\mathrm{i}kx}\,\mathrm{d}x \tag{1.6}$$

and note that $\hat{\phi}(k) = \hat{\phi}(-k)^*$, where the * denotes complex conjugation, because ϕ is real. Similarly, the intial condition $\phi_o(x)$ is written as

$$\phi_0(x) = \int \hat{\phi}_0(k)e^{\mathrm{i}kx}\mathrm{d}k. \tag{1.7}$$

Now plug the Fourier-integral representation (1.5) into (1.1) and one gets

$$\int \frac{\partial \hat{\phi}}{\partial t} e^{\mathrm{i}kx} \mathrm{d}k + \int \mathrm{i}kc\hat{\phi} e^{\mathrm{i}kx} \mathrm{d}k = 0 \tag{1.8}$$

$$\int \underbrace{\left[\frac{\partial \hat{\phi}}{\partial t} + \mathrm{i}kc\hat{\phi} \right]}_{=0} e^{\mathrm{i}kx} \mathrm{d}k = 0 \tag{1.9}$$

The quantity in brackets is the simple first-order ODE with constant, but *complex*, coefficients. Thus $\hat{\phi}(k)$ has solutions of the form $\hat{\phi}(k) = Ce^{-\mathrm{i}kct}$ and with the initial condition (1.7) at $t = 0$, it then is clear that the full solution is

$$\phi(x,t) = \int \hat{\phi}_0(k) e^{\mathrm{i}k(x-ct)} \mathrm{d}k$$

which is back to the old familiar form of $\phi_0(x - ct)$ a solution which just translates.

This is a fairly simple example of using Fourier analysis and substitution to get solutions to PDEs. However, this is a mainstay of wave analysis.

1.1.3 Relationship to Conservation Equations

In fluid dynamics, a conserved quantity - for example T with units of [stuff] where stuff could be °C, μg/L, etc. - obeys a conservation equation of the form

$$\frac{\partial T}{\partial t} + \nabla \cdot \mathbf{F}_T = 0 \tag{1.10}$$

where $\mathbf{F}_T$ is the vector flux of T. Flux is often written as a velocity × quantity, that is $\mathbf{F}_T = \mathbf{u}T$ with units [stuff m/s].

In one-dimension, (1.10) becomes

$$\frac{\partial T}{\partial t} + \frac{\partial uT}{\partial x} = 0.$$

Now if you let u be constant and pull it outside the equation one gets

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0.$$

which is the 1D wave equation with solutions of propagating waves of permanent form. We will see again that this 1D wave equation forms a kind of backbone for all sorts of conserved quantities (energy or wave action) in wave systems.

1.2 The Real Wave Equation: Second-order wave equation

Here, we now examine the second order wave equation, which appears many disciplines such as acoustics, optics, E&M, and geophysics. A variety of ocean waves follow this wave equation to a greater or lesser degree. The full second order wave equation is

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0 \tag{1.11}$$

where ∇^2 is the Laplacian operator operating in one, two, or three dimensions. Here again c is real and is constant. For analysis purposes, we restrict ourselves to the one-dimensional wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = 0. \tag{1.12}$$

on an infinite domain ($-\infty < x < \infty$). How many initial conditions does this equation require? Two - as it is a second order equation in time. Typically this means prescribing

$$\phi(x, t = 0) = a(x) \tag{1.13}$$

$$\frac{\partial}{\partial t}\phi(x, t = 0) = b(x) \tag{1.14}$$

If we were not considering an infinite domain, we would also need two boundary conditions.

1.2.1 General Solution

The general solution of (1.11) is

$$\phi(x, t) = f(x - ct) + g(x + ct) \tag{1.15}$$

that is two permanent forms propagating right and left. We can show this the same way as with the first-order wave equation with the variables $u = x - ct$ and $v = x + ct$, then $f(x - ct) = f(u)$ and $g(x + ct) = g(v)$ and

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = c^2 \frac{\mathrm{d}^2 f}{\mathrm{d}u^2} - c^2 \frac{\mathrm{d}f}{\mathrm{d}u} = c^2 \frac{\mathrm{d}^2 g}{\mathrm{d}v^2} - c^2 \frac{\mathrm{d}g}{\mathrm{d}v} = 0 \tag{1.16}$$

As (1.11) is linear, one can also linearly superimpose solutions.

1.2.2 Solution to Initial Value Problem: D'Alembert's Solution

We have the general solution (1.15) with initial conditions (1.13-1.14). At $t = 0$,

$$f(x) + g(x) = a(x) \tag{1.17}$$

$$-cf'(x) + cg'(x) = b(x) = c(g' - f') \tag{1.18}$$

where the prime denotes the derivative. The second equation can be integrated to

$$g(x) - f(x) = c^{-1} \int_{x_0}^{x} b(\tilde{x}) d\tilde{x} \tag{1.19}$$

where $\tilde{x}$ is a dummy variable of integration. Adding and subtracting (1.17) and (1.19) gives

$$f(x) = \frac{1}{2}\left[a(x) - c^{-1} \int_{x_0}^{x} b(\tilde{x}) d\tilde{x}\right] \tag{1.20}$$

$$g(x) = \frac{1}{2}\left[a(x) + c^{-1} \int_{x_0}^{x} b(\tilde{x}) d\tilde{x}\right] \tag{1.21}$$

which now determines that full solution

$$f(x-ct) = \frac{1}{2}\left[a(x-ct) - c^{-1} \int_{x_0}^{x-ct} b(\tilde{x}) d\tilde{x}\right] \tag{1.22}$$

$$g(x+ct) = \frac{1}{2}\left[a(x+ct) + c^{-1} \int_{x_0}^{x+ct} b(\tilde{x}) d\tilde{x}\right] \tag{1.23}$$

and rewritting

$$\int_{x_0}^{x-ct} b(\tilde{x}) d\tilde{x} = -\int_{x-ct}^{x_0} b(\tilde{x}) d\tilde{x}$$

yields the full solution of

$$\phi(x,t) = \frac{1}{2}\left[a(x-ct) + a(x+ct) + c^{-1} \int_{x-ct}^{x+ct} b(\tilde{x}) d\tilde{x}\right] \tag{1.24}$$

1.2.3 Plane wave solutions

Now we try to plug into (1.12) a plane wave propagating in the $+x$ direction of the form (1.4)

$$\phi = e^{\mathrm{i}(kx-\omega t)} \tag{1.25}$$

where $\omega > 0$, resulting in

$$[-\omega^2 + c^2k^2]e^{\mathrm{i}(kx-\omega t)} \implies \omega^2 = c^2k^2.$$

This has analogy to $f(x-ct)$ being a solution. What about

$$\phi = e^{\mathrm{i}(kx+\omega t)} \tag{1.26}$$

which results in

$$[-\omega^2 + c^2k^2]e^{\mathrm{i}(kx-\omega t)} \implies \omega^2 = c^2k^2.$$

Analogous to $g(x+ct)$ being a solution. They are the same! Here, the dispersion relationship is also **non-dispersive** as $|\omega|/k = c$ is a constant. Thus, all frequencies and wavenumbers propagate at the same speed - and if you know c and k, you also know the frequency.

1.2.4 Fourier Solution

The tactic here is to reduce the partial differential equation (1.11) to an ordinary differential equation via Fourier subsitution. We can now write a simple solution to $\phi(x,t)$ via Fourier analysis. Here, for simplicity, we assume that the initial condition for $\partial\phi/\partial t = b(x) = 0$. Repeating the analysis for the first-order wave equation, we write $\phi(x,t)$ as a Fourier integral,

$$\phi(x,t) = \int \hat{\phi}(k,t)e^{\mathrm{i}kx}\mathrm{d}k \tag{1.27}$$

and similarly the intial condition $a(x)$ is written as

$$a(x) = \int \hat{a}(k)e^{\mathrm{i}kx}\mathrm{d}k. \tag{1.28}$$

Now plug the Fourier-integral representation (1.5) into (1.12) and one gets

$$\int \frac{\partial^2\hat{\phi}}{\partial t^2}e^{\mathrm{i}kx}\mathrm{d}k + \int c^2k^2\hat{\phi}e^{\mathrm{i}kx}\mathrm{d}k = 0 \tag{1.29}$$

$$\int \underbrace{\left[\frac{\partial^2\hat{\phi}}{\partial t^2} + c^2k^2\hat{\phi}\right]}_{=0} e^{ikx}\mathrm{d}k = 0 \tag{1.30}$$

The quantity in brackets is the good-old fashioned harmonic oscillator equation. It has solutions

$$\hat{\phi}(k,t) = A(k)e^{\mathrm{i}\omega t} + B(k)e^{-im\omega t} \tag{1.31}$$

where $\omega^2 = c^2k^2$, as the dispersion relationship comes up again! Now since $\partial\phi/\partial t = 0$ at $t = 0$ this implies that

$$\mathrm{i}\omega[A(k) - B(k)] = 0 \implies A(k) = B(k), \tag{1.32}$$

thus the full Fourier solution is

$$\phi(x,t) = \frac{1}{2}\int \hat{a}(k)[e^{\mathrm{i}k(x-ct)} + e^{\mathrm{i}k(x+ct)}]\mathrm{d}k \tag{1.33}$$

which is back to the familiar form of $f(x-ct)$ and $g(x+ct)$ solutions. One can also do this analysis for a general initial condition, but it is just more algebra.

1.2.5 Boundary Conditions: Standing Waves

Up to now, only infinite spatial domains have been considered and thus all waves are **progressive**. Now instead of an infinite domain, consider a finite domain with boundary conditions of $\phi = 0$ are applied at $x = 0$ and $x = L$ - recall that as wave equation is second order in space, two boundary conditions are required. The same initial conditions as above are used, $\phi(x, t = 0) = a(x)$ and $\partial\phi(x, t = 0)/\partial t = 0$.

There are formal methods for solving (1.12) on $0 < x < L$ that involve separation of variables. Basically, you assume a solution $\phi(x, t) = T(t)X(x)$ - which is typically covered in undergraduate mathematical physics classes. Here, we will use a bit of intuition and Fourier. From the boundary conditions, one could expect $\phi \propto \sin(n\pi x/L)$ where n is a positive definite integer because with sin(), $\phi = 0$ at $x = 0$ and $x = L$. Propose a solution of the form

$$\phi(x, t) = \sum_{n=1}^{\infty} \hat{\phi}_n(t) \sin(n\pi x/L). \tag{1.34}$$

Why a sum and not an integral? Because only particular sin() that match the boundary conditions are allowed. On an infinite domain, we use integrals instead of sums. The initial conditions must also be similarly written

$$\phi(x, t = 0) = a(x) = \sum_{n=1}^{\infty} \hat{a}_n \sin(n\pi x/L).$$

Plugging into the wave-equation (1.12) we get

$$\sum_{n=1}^{\infty} \underbrace{\left[\frac{\partial^2 \hat{\phi}_n}{\partial t^2} + \frac{n^2\pi^2c^2}{L^2}\hat{\phi}_n\right]}_{=0} \sin(n\pi x/L) = 0$$

The quantity in brackets is again the harmonic oscillator equation with solution

$$\hat{\phi}_n(t) = c_{1n} \cos(n\pi ct/L) + c_{2n} \sin(n\pi ct/L)$$

Now recall that $\partial\phi/\partial t = 0$ at $t = 0$ which means that all $c_{2n} = 0$. Matching the initial condition implies that $c_{1n} = \hat{a}_n$ and thus the full solution is written as

$$\phi(x, t) = \sum_{n=1}^{\infty} \hat{a}_n \cos(n\pi ct/L) \sin(n\pi x/L). \tag{1.35}$$

Now, this solution is interesting. It is not obviously in a form of $f(x - ct)$ and $g(x + ct)$ as were the earlier solutions on an infinite domain. Such solutions are **progressive** in that the waves move. Instead the solution has the form $\cos(ct)\sin(x)$ - representing harmonic motions that is standing in space and time. Such waves are known as **standing** waves. Think of a guitar string with modes $n = 1$, $n = 2$, etc.

How to think about the distinction between progressive and standing? Departing from exact mathematical solutions and using heuristic thinking, suppose you have a wave train $+x$

propagating towards a perfectly reflective wall at $x = L$. A little later, there will be waves propagating towards and away from the wall, *i.e.*, $\phi = \cos(x - ct) + \cos(x + ct)$ (supposing $k = 1$). Using trigonometric identies[1] we can rewrite

$$\phi = \cos(x - ct) + \cos(x + ct) = \cos(ct)\cos(x).$$

Thus, we see that standing waves are superposition of waves propagating in the $\pm x$ direction at the same frequency and wavenumber. This provides an interpretation of the finite domain solution. There really are $f(x - ct)$ and $g(y + ct)$ modes propagating within $0 < x < L$, but they are constantly reflecting back and forth.

Many wave types allow for both **progressive** and **standing** components. In summary, **progressive** waves have a form $\cos(kx - \omega t)$ or $\exp(\mathrm{i}kx - \omega t)$. **Standing** waves have a form akin to $\cos(kx)\sin(\omega t)$.

[1] $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

1.3 Problem Set

1. Imagine you are lowering a thermistor (temperature sensor) to measure the vertical (z) temperature profile. You don't have a pressure sensor on the instrument but you know that the instrument falls at a steady velocity w [m/s]. From the time-series of temperature that the thermistor measures, how would you estimate the $T(z)$?

2. Consider the two PDEs

$$\phi_t + \gamma\phi_{xxx} = 0 \tag{1.36}$$

$$\phi_t + \gamma\phi_{xxxx} = 0 \tag{1.37}$$

   For each, determine (a) if plane wave solutions (*i.e.*, $\exp[\mathrm{i}(kx-\omega(k)t])$ are permissible or if not why not, and (b) if so, what is the dispersion relationsihp $\omega = \omega(k)$.

3. For a general PDE with

$$\frac{\partial^n \phi}{\partial t^n} + c\frac{\partial^m \phi}{\partial x^m} = 0,$$

   (a) What property do n and m have to have to allow plane wave solution? (b) What is the resulting dispersion relationship?

4. Consider the 3D second-order wave equation (1.11) with plane wave solutions $\propto \exp[\mathrm{i}(kx + ly + mz)]$. What is the dispersion relation $\omega = \omega(k, l, z)$?

5. For the 1D 2nd-order wave equation on $-\infty < x < \infty$, consider the initial condition $\phi(x, t = 0) = \delta(x)$, where $\delta(x)$ is the Dirac delta function, and $\partial\phi(x, t = 0)/\partial t = 0$, what is the solution for $\phi(x, t)$?

6. (EXTRA CREDIT) Similar to above but with initial conditions $\phi(x, t = 0) = 0$ and $\partial\phi(x, t = 0)/\partial t = \delta(x)$, what is the solution for $\phi(x, t)$? There are two approaches

   (a) Use the D'Alembert's solution (1.24).

   (b) Use the Fourier approach: Use the fact that

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}kx}\mathrm{d}k$$

   and

   i. Use the initial conditions to come up with an expression for $A(k)$ and $B(k)$ in (1.31).

   ii. Which are larger in amplitude, longer or shorter wavelengths?

(c) Can you show that these two solutions are identical?

7. Consider the 2D 2nd-order wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2} \right) = 0. \tag{1.38}$$

in a rectangular domain $0 < x < L$ and $0 < y < L$ with boundary condition of $\phi = 0$ on all sides and $\partial\phi/\partial t = 0$ at $t = 0$. Use your intuition and the results from the 1D section (1.35), what would the form be for a solution for $\phi(x, y, t)$? Plug it into (1.38), does it work as a solution?

Chapter 2

Linear Surface Gravity Waves A

2.1 Definitions

Here we define a number of wave parameters and give their units for the surface gravity wave problem:

- wave amplitude a : units of length (m)
- wave height $H = 2a$: units of length (m)
- wave radian frequency ω : units of rad/s
- wave frequency $f = \omega/(2\pi)$: units of 1/s or (Hz)
- wave period T - time between crests: $T = 1/f$: units of time (s)
- wavelength λ - distance between crests : units of length (m)
- wavenumber $k = 2\pi/\lambda$: units of rad/length (rad/m)
- phase speed $c = \omega/k = \lambda/T$: units of length per time (m/s)
- group speed $c_g = \partial\omega/\partial k$: units of length per time (m/s)
- Typically the wave propagates in the horizontal direction $+x$.
- The vertical coordinate is z with $z = 0$ at the still water surface and increasing upwards.

2.2 Potential Flow

Here we assume that readers have a basic understanding of fluid dynamics and particularly (irrotational) potential flow. Here we review where irrotational (*i.e.,* zero vorticity) flow comes from. Consider first a perfect fluid with gravity but without stratification or rotation. The governing continuity and momentum equations are

$$\nabla \cdot \mathbf{u} = 0 \tag{2.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \rho^{-1} \nabla p - g\mathbf{k} \tag{2.2}$$

where p is pressure, ρ is a constant density, and $\mathbf{u}$ is the velocity vector field. Taking the curl of the momentum equation yields (review from fluids)

$$\frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta + \zeta \cdot \nabla \mathbf{u} = \frac{\mathrm{D}\zeta}{\mathrm{D}t} + \zeta \cdot \nabla \mathbf{u} = 0 \tag{2.3}$$

where ζ is the vector vorticity $\zeta = \nabla \times \mathbf{u}$. So if $\zeta(\mathbf{x}, t) = 0$ everywhere at time $t = 0$, then $\zeta(\mathbf{x}, t) = 0$ for all time. Note that vorticity is also often represented with the symbol ω.

A general vector field can be written as a sum of a potential component and a rotational component, that is

$$\mathbf{u} = \nabla \phi + \nabla \times \psi \tag{2.4}$$

where ϕ is a velocity potential and ψ is a streamfunction. If $\nabla \times \mathbf{u} = 0$, then $\nabla^2 \psi = 0$ and $\psi = 0$. Thus the velocity field reduces to $\mathbf{u} = \nabla \phi$. With the continuity equation $\nabla \cdot \mathbf{u} = 0$, one gets that

$$\boxed{\nabla^2 \phi = 0} \tag{2.5}$$

for all $\mathbf{x}$ and t. This type of zero-vorticity flow is called irrotational flow or potential flow.

2.3 Statement of the classic problem

The derivation here for linear surface gravity waves follows that of Kundu (Chapter 7), but is found in many other places as well. Here we set up the classic full surface gravity wave problem which we assume is a wave that

- plane waves propagating in the $+x$ direction only.
- The sea-surface η is a function of x and time t : $\eta(x, t)$
- Waves propagating on a flat bottom of depth h.

Thus water velocity is 2D and is due to a velocity potential ϕ

$$\mathbf{u} = (u, 0, w) = \nabla \phi$$

which with continuity implies that in the interior of the fluid

$$\nabla^2 \phi = 0. \tag{2.6}$$

Next a set of boundary conditions are required in order to solve (2.6). These classic boundary conditions are

1. No flow through the bottom: $w = \partial\phi/\partial z = 0$ at $z = -h$.
2. Surface kinematic: particles stay at the surface: $D\eta/Dt = w$ at $z = \eta(x, t)$.
3. Surface dynamic: surface pressure p is constant on the water surface or $p = 0$ at $z = \eta(x, t)$. This couples velocity and η at the sea surface through Bernouilli's equation that applies to irrotational flow.

The solution to (2.6) with the boundary conditions is a statement of the exact problem for irrotational nonlinear surface gravity waves on an arbitrary bottom. As such it includes a lot of physics including wave steepening, the onset of overturning, reflection, etc. There are models that solve (2.6) with these boundary conditions exactly. This does not include dissipative process such as full wave breaking, wave dissipation due to bottom boundary layers, etc., as friction has been neglected here.

2.4 Simplifying Boundary Conditions: Linear Waves

Boundary conditions #2 and #3 are complex as they are evaluated at a moving surface and thus they need to be simplified. It is this simplification that leads to solutions for *linear* surface gravity waves. This derivation can be done formally for a small non-dimensional parameter. For deep water this small non-dimensional parameter would be the wave steepness ak, where a is the wave amplitude and k is the wavenumber. Here, the derivation will be done loosely and any terms that are *quadratic* will simply be neglected.

2.4.1 Surface Kinematic Boundary Condition

Lets start with the #2 the surface kinematic boundary condition,

$$\frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = w\Big|_{z=\eta}. \tag{2.7}$$

Neglecting the quadratic term and writing $w = \partial\phi/\partial z$, we get the simplified and linear equation

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}\Big|_{z=\eta}. \tag{2.8}$$

However, the right-hand-side of (2.8) is still evaluated at the surface $z = \eta$ which is not convenient. This is still not easy to deal with. So a Taylor series expansion is applied to

$\partial\phi/\partial z$ so that

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=\eta} = \left.\frac{\partial\phi}{\partial z}\right|_{z=0} + \eta\left.\frac{\partial^2\phi}{\partial z^2}\right|_{z=0} \tag{2.9}$$

Again, neglecting the quadratic terms in (2.9), we arrive at the fully linearized surface kinematic boundary condition

$$\frac{\partial\eta}{\partial t} = \left.\frac{\partial\phi}{\partial z}\right|_{z=0} \tag{2.10}$$

2.4.2 Surface Dynamic Boundary Condition

The surface dynamic boundary condition stating that pressure is constant (or zero) along the surface is a nice simple statement. However, the question is how to relate this to the other variables we are using namely η and ϕ. In irrotational motion, Bernoulli's equation applies

$$\left.\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz = 0\right|_{z=\eta} \tag{2.11}$$

where ρ is the (constant) water density and g is gravity. Again, quadratic terms will be neglected and if $p = 0$ this equation reduces to

$$\left.\frac{\partial\phi}{\partial t} + g\eta = 0\right|_{z=\eta} \tag{2.12}$$

This boundary condition appears simple but again the term $\partial\phi/\partial t$ is applied on a moving surface η, which is a mathematical pain. Again a Taylor series expansion can be applied

$$\left.\frac{\partial\phi}{\partial t}\right|_{z=\eta} = \left.\frac{\partial\phi}{\partial t}\right|_{z=0} + \eta\left.\frac{\partial^2\phi}{\partial t\partial z}\right|_{z=0} \simeq \left.\frac{\partial\phi}{\partial t}\right|_{z=0} \tag{2.13}$$

once quadratic terms are neglected.

2.4.3 Summary of Linearized Surface Gravity Wave Problem

$$\nabla^2\phi = 0 \tag{2.14a}$$

$$\frac{\partial\phi}{\partial z} = 0, \text{ at } z = -h \tag{2.14b}$$

$$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial z}, \text{ at } z = 0 \tag{2.14c}$$

$$\frac{\partial\phi}{\partial t} = -g\eta, \text{ at } z = 0 \tag{2.14d}$$

Now the question is how to solve these equations and boundary conditions. The answer is the time-tested one. Plug in a solution, in particular for this case, plug in a *propagating wave*

2.5 Solution to the Linearized Surface Gravity Wave Problem

Here we start off assuming a solution for the surface of a plane wave with amplitude a traveling in the $+x$ direction with wavenumber k and radian frequency ω. This solution for $\eta(x,t)$ looks like

$$\eta = a\cos(kx - \omega t) \tag{2.15}$$

Next we assume that ϕ has the same form in x and t, but is separable in z, that is

$$\phi = f(z)\sin(kx - \omega t) \tag{2.16}$$

Thus we can write

$$\nabla^2\phi = \left[\frac{d^2 f}{dz^2} - k^2 f\right]\sin(\ldots) = 0.$$

The term in [] must be zero identically thus,

$$\frac{d^2 f}{dz^2} - k^2 f = 0,$$

which as a linear 2nd order constant coefficient ODE has solutions of

$$f(z) = Ae^{kz} + Be^{-kz}$$

and by applying the bottom boundary condition $\partial\phi/\partial z = df/dz = 0$ at $z = -h$ leads to

$$B = Ae^{-2kh},$$

Note that this is the first appearance of kh. However we still need to know what A is. Next we apply the surface kinematic boundary condition (2.10)

$$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial z}, \quad \text{at} \quad z = 0$$

which results in

$$a\omega\sin(\ldots) = k(A - B)\sin(\ldots)$$

which give A and B. This leads to a expression for ϕ of

$$\phi = \frac{a\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}\sin(kx - \omega t) \tag{2.17}$$

So we almost have a full solution, the only thing missing is that for a given a and a given k, we don't know what the radian frequency ω should be. Another way of saying this is that we don't know the dispersion relationship. This is gotten by now using the surface dynamic boundary condition by plugging (4.14) and (2.15) into (2.12) and one gets

$$\left[-\frac{a\omega^2}{k}\frac{\cosh(kh)}{\sinh(kh} = -ag\right]\cos(\ldots)$$

which simplifies to the classic linear surface gravity wave dispersion relationship

$$\omega^2 = gk\tanh(kh) \tag{2.18}$$

The pressure under the fluid is can also be solved for now with the linearized Bernoulli's equation: $p = -\rho gz - \rho\partial\phi/\partial t$. This leads to a the still (or *hydrostatic*) pressure ($-\rho gz$) and wave part of pressure $p_w = -\rho\partial\phi/\partial t$.

The full solution for all variables is

$$\eta(x,t) = a\cos(kx-\omega t) \tag{2.19a}$$

$$\phi(x,z,t) = \frac{a\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}\sin(kx-\omega t) \tag{2.19b}$$

$$u(x,z,t) = a\omega\frac{\cosh[k(z+h)]}{\sinh(kh)}\cos(kx-\omega t) \tag{2.19c}$$

$$w(x,z,t) = a\omega\frac{\sinh[k(z+h)]}{\sinh(kh)}\sin(kx-\omega t) \tag{2.19d}$$

$$p_w(x,z,t) = \frac{\rho a\omega^2}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}\cos(kx-\omega t) \tag{2.19e}$$

2.6 Implications of the Dispersion Relationship

The dispersion relationship is

$$\omega^2 = gk\tanh(kh)$$

and is super important. To gain better insight into this, one can non-dimensionalize ω by $(g/h)^{1/2}$ so that

$$\frac{\omega^2 h}{g} = f(kh) = kh\tanh(kh), \tag{2.20}$$

where the nondimensional parameter kh is all important. It can be thought of as a nondimensional depth or as the ratio of water depth to wavelength. To examine the limits of small and large kh, we first review $\tanh(x)$,

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{2.21}$$

and so for small x, $\tanh(x) \simeq x$ and for large x, $\tanh(x) \simeq 1$.

Here we define *deep* water as that were the water depth h is far larger than the wavelength of the wave λ, *i.e.,* $\lambda/h \ll 1$ which can be restated as $kh \gg 1$. With this $\tanh(kh) = 1$ and the dispersion relationship can be written as

$$\frac{\omega^2 h}{g} = kh, \Rightarrow \omega^2 = gk \tag{2.22}$$

with wave phase speed of

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}} \tag{2.23}$$

Similarly, *shallow* water can be defined as where the depth h is much smaller than a wavelength λ. This means that $kh \ll 1$, which implies that $\tanh(kh) = kh$ and the dispersion relationship simplifies to

$$\frac{\omega^2 h}{g} = (kh)^2, \Rightarrow \omega^2 = (gh)k^2 \Rightarrow \omega = (gh)^{1/2}k \tag{2.24}$$

and the wave phase speed

$$c = \frac{\omega}{k} = \sqrt{gh} \tag{2.25}$$

2.7 Nondimensionalization and Linearization

Here we now examine how good or bad the ad-hoc linearization of the full kinematic boundary condition (2.7) which led to (2.10) is and what it depends upon using the linear theory solutions (2.19) for *deep* water. First clearly the appropriate scale of time t is ω^{-1} and for space (x, z) is k^{-1} (only in deep water). If we now take (2.7), and scale the equation we get the following (neglected terms are canceled out)

$$\begin{aligned} \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \cancelto{}{u\frac{\partial \eta}{\partial x}} = w\Big|_{z=\eta} . \\ a\omega + a^2\omega k = a\omega\big|_{z=\eta} \\ a\omega\left[1 + \cancelto{}{(ak)} = 1\right]_{z=\eta} \end{aligned}$$

so we see that the neglected term was order ak relative to the other terms. If we go to the next step of Taylor series expanding about $z = 0$ (2.9), we get

$$\begin{aligned} \frac{\partial \phi}{\partial z}\Big|_{z=\eta} = \frac{\partial \phi}{\partial z}\Big|_{z=0} + \cancelto{}{\eta\frac{\partial^2 \phi}{\partial z^2}}\Big|_{z=0} \\ = a\omega + a^2\omega k \\ = a\omega(1 + \cancelto{}{ak}) \end{aligned}$$

so we see here that the higher order Taylor series term is *also* order ak relative to the other terms. This means that we are neglecting terms of order ak relative to the other terms. What is ak? It is the non-dimensional slope, ie a wave amplitude divided by a wavelength. What does this mean? It means that in order for linear theory to be valid the slope $ak \ll 1$. Does that seem reasonable to you? Lastly, the full problem equations could've been nondimensionalized from the beginning which judicious choice of velocity, time, and length-scales and terms collected. In deep water, the wave slope ak would come out. Here our treatment will be linear, but it is possible to write $\epsilon = ak$ and go on to higher order to get nonlinear wave solutions.

2.8 Problem Set

1. In $h = 1$ m and $h = 10$ m water depth, what frequency $f = \omega/(2\pi)$ (in Hz) corresponds to $kh = 0.1$, $kh = 1$, and $kh = 10$ from the full dispersion relationship? Make a 3 by 2 element table.

2. Plot the non-dimensional dispersion relationship $\omega^2 h/g$ versus kh. Then plot the shallow water approximation to this (2.24). At what kh is the shallow water approximation in 20% error?

3. The shallow water approximation to the non-dimensional dispersion relationship (2.24) is $\omega^2 h/g = (kh)^2$. Derive the next higher order in kh dispersion relationship from the full dispersion relationship $\omega^2 h/g = kh \tanh(kh)$. What is the corresponding phase speed c?

4. In the shallow water approximation:

   (a) Write out the expression for u as a function of a, h, and c.

   (b) What non-dimensional parameter comes out of the ratio of u/c?

   (c) What limitations on size does this parameter have?

5. In 5-m water depth and a wave of period $T = 18$ s and wave height $H = 2a = 1$ m.

   (a) Do you think that shallow water approximation is valid? Based on results from questions above?

   (b) What is the magnitude of u?

6. In water depth h, suppose pressure p_w (2.19e) and horizontal velocity u (2.19c) are measured at the same vertical location z. Derive an expression for $p_w/(\rho u)$.

7. In water depth h, suppose a pressure sensor measures pressure p_w (2.19e) at vertical location z_p and a current meter measures u (2.19c) at a different vertical location z_u. In real life, this a often the case.

   (a) Given $p_w(z_p, t)$, give an expression for wave pressure at z_u.

   (b) Then write an expression for the ratio of $(p_w)/(\rho u)$ at z_u using p_w measured at z_p and u measured at z_u.

8. Using the scalings for the linear solutions (2.19a –2.19e), non-dimensionalize the dynamic boundary condition (2.11) and the Taylor series expansion (2.12). What nondimensional parameter must be small for the linear approximation to be valid? *i.e.,* to justify the neglect of $|\nabla\phi|^2$ in (2.11) and the neglect of $\eta\partial^2\phi/\partial t\partial z$ in (2.12)?

Chapter 3

Linear Surface Gravity Waves B, Energy Conservation, Energy Flux, and Group Velocity

Here, mean properties of the linear surface gravity wave field will be considered. These properties include wave energy and wave energy flux. Other mean properties such as wave mass flux, also known as *Stokes drift*, and wave momentum fluxes will not be considered here. Some of these wave properties will be depth averaged and others will not be, so keep that in mind.

On a practical level it is worth considering the potential challenges in modeling waves on a global level. Surface gravity waves in the ocean typically have periods between 3–20 s. The longer waves (say periods longer than 12 s) are classified as *swell* and shorter waves (say less than 8 s) classified as *sea*. For swell in deep water ($\omega^2 = gk$), a typical scale for the wavelength is $\lambda \approx 100$ m. In order to numerically simulate this with the equations of the previous chapter, one might think you need perhaps 10 grid points to resolve a cos(), corresponding to a grid spacing of $\Delta = 10$ m. To do a 1000 km by 1000 km domain (which is already much smaller than say the North Atlantic), this implies that one would require 10^{10} grid points. This is huge and beyond the capability of even the fastest supercomputers. Furthermore, the same argumens apply to the time-scale. So in order to do basin-wide wave modeling, one needs a different set of equations. These equations are based on wave energy conservation nad will be derived in basic form here.

There is another reason to consider wave energetics and that is because just as with say a pendulum, considering energetics leads to greater insight into the system.

3.1 Wave Energy

Specific energy is the energy per unit volume, and has units $\mathrm{J\,m^{-3}}$ so that the specific energy integrated over a volume has units of J. We will use this concept to think of wave energy as the depth-integrated specific energy. As such it should then have units of $\mathrm{J\,m^{-2}}$ so that by averaging wave-energy over an area, one gets Joules (J). We will also think of the wave energy as a time-averaged or mean quantity, where the time-average is defined as the average energy of waves over a wave period.

Wave energy E can be though of as the sum of kinetic (KE) and potential (PE) energy, $E = \mathrm{KE} + \mathrm{PE}$. Lets first calculate the potential energy (PE), defined as the excess potential energy due to the wave field. Thus the instantaneous potential energy is

$$\rho g \left[\int_{-h}^{\eta} z \,\mathrm{d}z - \int_{-h}^{0} z \,\mathrm{d}z \right] = \rho g \int_{0}^{\eta} z \,\mathrm{d}z = \frac{1}{2} \rho g \eta^2 = \frac{1}{2} \rho g a^2 \cos^2(\omega t). \tag{3.1}$$

Now we time-average (3.1) over a wave period, with the identity that $(1/T) \int_0^T \cos^2(\omega t) dt = 1/2$, we get the mean potential energy PE,

$$\mathrm{PE} = \frac{1}{4} \rho g a^2. \tag{3.2}$$

Next we consider the kinetic energy. The local kinetic energy per unit volume is $(1/2)\rho|\mathbf{u}|^2$, and so depth-integrated this becomes

$$\frac{1}{2} \rho \int_{-h}^{\eta} |\mathbf{u}|^2 \,\mathrm{d}z = \rho \int_{-h}^{\eta} (u^2 + w^2) \,\mathrm{d}z. \tag{3.3}$$

However, here we are interested in the *linear* kinetic energy, *i.e.,* that kinetic energy which is appropriate to linear theory. As linear theory is correct to $O(\epsilon)$ where in deep water $\epsilon = ak$ then kinetic (and potential energy) should be correct to $O(\epsilon^2)$. As u^2 is already a $O(\epsilon^2)$ quantity, we do not need to vertically integrate all the way to $z = \eta$ but can stop at $z = 0$ as this would only add another power of ϵ to the kinetic energy estimate. That is

$$\int_{0}^{\eta} (u^2 + w^2) \,\mathrm{d}z \approx \eta (u^2 + w^2) \approx O(\epsilon^3).$$

Recall that this is not a *formal* approach, only a heuristic one. Thus, for linear waves

$$\rho \int_{-h}^{\eta} (u^2 + w^2) \,\mathrm{d}z \simeq \rho \int_{-h}^{0} (u^2 + w^2) \,\mathrm{d}z. \tag{3.4}$$

Using the solutions (2.19c and 2.19d) and depth-integrating and time-averaging over a wave-period one gets

$$\mathrm{KE} = \frac{1}{4} \rho g a^2. \tag{3.5}$$

The first thing to note is that the the kinetic and potential energy are the same (KE = PE), that is the wave energy is *equipartioned.* This is a fundamental principle in all sorts of linear wave systems. But that is not a topic for here.

Now consider the total mean wave energy E,

$$E = \text{KE} + \text{PE} = \frac{1}{2}\rho g a^2 \tag{3.6}$$

Now if one defines the wave height $H = 2a$, then the wave energy is written as

$$E = \frac{1}{8}\rho g H^2. \tag{3.7}$$

Note this also can be more generally written as

$$E = \rho g \overline{\eta^2} \tag{3.8}$$

where $\overline{\eta^2}$ is the variance of the sea-surface elevation. Thus wave energy can be linked to wave amplitude variance. This also allows wave energy spectra to be calculated from sea surface elevation spectra.

3.2 A Digression on Fluxes 2.

A local flux is a quantity $\times$ velocity, so it should have units of Q m/s. For example,

- temperature flux: $T\mathbf{u}$
- mass flux: $\rho\mathbf{u}$
- volume flux: $\mathbf{u}$

Transport is the flux through an area A. So this has units of $Q \times \text{m}^3\text{s}^{-1}$ and transport T_Q can be written as

$$T_Q = \int (\mathbf{u}\cdot\mathbf{n}) Q \, dA,$$

where $\mathbf{n}$ is the outward unit normal. An example of volume transport can be the transport of the Gulf Stream ≈ 100 Sv where a Sv is $10^6\ \text{m}^3\,\text{s}^{-1}$. Or consider transport from a faucet of 0.1 L/s. A liter is $10^{-3}\ \text{m}^3$ so this faucet transport is $10^{-4}\ \text{m}^3\,\text{s}^{-1}$. Thus, a liter jar is filled in 10 s. If the faucet area is 1 cm^2, then the water velocity in the faucet is $1\ \text{m}\,\text{s}^{-1}$. A heat flux example is useful to consider. For example heat content per unit volume is $\rho c_p T$, where c_p is the specific heat capacity with units Jm^{-3}. This implies that by integrating over a volume, one gets the heat content (thermal energy or internal energy) which has units of Joules. So the local heat flux is $\rho c_p T\mathbf{u}$ (c_p is the specific heat) which then has units of Wm^{-2}. When integrated over an area,

$$\int \rho c_p T(\mathbf{u}\cdot\mathbf{n})\,dA \tag{3.9}$$

gives units of Watts (W).

Knowing flux is important for many things practical and engineering. However, one fundamental property of flux is its role in a tracer conservation equation. A conserved tracer ϕ evolves according to

$$\frac{\partial\phi}{\partial t} + \nabla\cdot\mathrm{Flux} = 0, \tag{3.10}$$

so that the divergence ($\nabla\cdot()$) of the flux gives the rate of change. This base equation can describe many things from traffic jams to heat evolution in a pipe to the Navier-Stokes equations. *What happens if the tracer is not conserved?*

A key point to the flux is that through the divergence theorem, the volume integral of ϕ evolves according to,

$$\frac{d}{dt}\int_V \phi dV = \int_{\partial V} \mathbf{F}\cdot\hat{n} dA$$

where the area-integrated flux $\mathbf{F}$ into or out of the volume gives the rate of change. This concept is useful in many physical problems including those with waves!

Depth-integrated Fluxes

Here, with monochromatic waves propagating in the $+x$ direction, we will typically consider fluxes (but not always) through the yz plane. This means that the normal to the plane $\hat{n}$ is in the $+x$ direction, and that $\mathbf{u}\cdot\hat{n} = u$, the component of velocity in the $+x$ direction. This makes the depth integrated flux of quantity Q

$$\int Qu\,\mathrm{d}z$$

with units $Q\mathrm{m}^2\,\mathrm{s}^{-1}$.

3.3 Wave Energy Flux and Wave Energy Equation

Now we calculate the wave energy flux. The starting point is the conservation equation for momentum, which here are the inviscid incompressible Navier-Stokes equations,

$$\nabla\cdot\mathbf{u} = 0 \tag{3.11a}$$

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = \nabla p - g\rho\hat{\mathbf{k}} \tag{3.11b}$$

where $\mathbf{k}$ is the unit vertical vector.

Now, as before we consider only the linear terms and thus we neglect the nonlinear terms $(\mathbf{u} \cdot \nabla \mathbf{u})$. Then an energy equation is formed by multiplying (3.11b) by $\rho \mathbf{u}$. The first terms becomes $(1/2)\partial |\mathbf{u}|^2/\partial t$ after integrating by parts. The pressure terms becomes $\mathbf{u} \cdot \nabla p = \nabla \cdot (\mathbf{u}p) - p \nabla \cdot \mathbf{u}$, and because the flow is incompressible ($\nabla \cdot \mathbf{u} = 0$) we are left with

$$\rho \frac{1}{2} \frac{\partial |\mathbf{u}|^2}{\partial t} = -\nabla \cdot (\mathbf{u}p) - g\rho w. \tag{3.12}$$

as $\mathbf{u} \cdot \hat{\mathbf{k}} = w$. We can move the gravity term over to the LHS and get,

$$\rho \left(\frac{1}{2} \frac{\partial |\mathbf{u}|^2}{\partial t} + gw \right) = -\nabla \cdot (\mathbf{u}p). \tag{3.13}$$

which is almost in the form of a conservation equation driven by a flux-divergence (3.10). Here the LHS can be thought of the time-derivative of the local kinetic and potential energies, respectively. On the RHS, the quantity $\mathbf{u}p$ is the *local* energy flux. Note that this does, sort of, look like a classic flux (velocity times quantity) with pressure having units of (Nm^{-2}) which is Jm^{-3}, which is energy per unit volume!

Lets first look at the energy-flux term. The depth-integrated and time-averaged wave energy flux F in the yz plane (*i.e.,* flux in the $+x$ direction) is

$$F = \left\langle \int_{-h}^{0} pu \, \mathrm{d}z \right\rangle. \tag{3.14}$$

The upper limit on the integral for (3.14) is $z = 0$ and not $z = \eta$ because this is the *linear* energy flux and assumes that η is small. In regards to linear theory, energy and energy flux are second order quantities, or said mathematically F is an $O(\epsilon^2)$ quantity and so terms of $O(\epsilon^3)$ or higher can be ignored. Higher order nonlinear theories can include the neglected kinetic energy component from $z = 0$ to $z = \eta$.

Now we just need to plug in the solutions and average and we get the wave energy flux. The pressure is the sum of the hydrostatic component $\bar{p}$ and the wave component p_w (2.19e). Because u (2.19c) is periodic and $\bar{p}$ is steady,

$$\left\langle \int_{-h}^{0} \bar{p} u \, \mathrm{d}z \right\rangle = 0 \tag{3.15}$$

leaving

$$F = \left\langle \int_{-h}^{0} p_w u \, \mathrm{d}z \right\rangle$$

Plugging in (2.19c) and (2.19e) and performing the integral results in

$$F = \frac{1}{2} \rho g a^2 \left[\frac{\omega}{k} \frac{1}{2} \left(1 + \frac{2kh}{\sinh(2kh)} \right) \right]$$

Now the wave energy flux can can be rearranged to look like

$$F = Ec\frac{1}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right) \tag{3.16}$$

which looks like a quantity (in this case E) times a type of velocity (here c) times a non-dimensional parameter $\star = (1/2)(1 + 2kh/\sinh(2kh))$. Lets consider two limits, deep water: $kh \to \infty$ then $\star \to 1$ and shallow water $kh \to 0$ gives $\star = 1/2$.

So perhaps one could redefine the velocity associated with the flux as c_g

$$c_g = c\frac{1}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right) \tag{3.17}$$

which we call the group velocity. Then the depth-integrated and time-averaged wave energy flux is

$$F = Ec_g \tag{3.18}$$

which is analogous to the flux definition (stuff times velocity) discussed earlier.

Now how is the group velocity related to the dispersion relationship $\omega^2 = gk\tanh(kh)$? Well first the wave phase speed is

$$c = \frac{\omega}{k} = \frac{[g\tanh(kh)]^{1/2}}{k^{1/2}} \tag{3.19}$$

and

$$\begin{aligned}\frac{\partial\omega}{\partial k} &= \frac{1}{2}\left[gk\tanh(kh)\right]^{-1/2}\left(g\tanh(kh) + gk\cosh^{-2}(kh)\right) &(3.20)\\ &= c\frac{1}{2}\left[1 + \frac{2kh}{\sinh(2kh)}\right]. &(3.21)\end{aligned}$$

So c_g, which is the velocity associated with the wave energy flux, is also

$$c_g = \frac{\partial\omega}{\partial k}. \tag{3.22}$$

This is rather interesting. This implies that wave energy propagates at a speed $\partial\omega/\partial k$ different from the speed at which wave crests propagate $c = \omega/k$. Is this a coincidence? This will be examined in the problem sets and subsequent chapter.

3.3.1 A Wave Energy Conservation Equation

Going back to the local energy equation (3.13)

$$\rho\left(\frac{1}{2}\frac{\partial|\mathbf{u}|^2}{\partial t} + \rho g w\right) = -\nabla\cdot(\mathbf{u}p), \tag{3.23}$$

we've already derived the depth-integrated wave energy flux (3.14)–(3.18) from the RHS of (3.23). Now, we can re-derive the kinetic and potential energy by depth-integrating the LHS of (3.23) and noting that for this linearized case $w = dz/dt$, so that

$$\int_{-h}^{\eta} \frac{\partial}{\partial t}\left(\frac{1}{2}\rho|\mathbf{u}|^2\right) dz + \int_{-h}^{\eta} \rho g w \tag{3.24}$$

$$= \frac{\partial}{\partial t}\left(\int_{-h}^{0} \frac{1}{2}\rho|\mathbf{u}|^2\, dz + \int_{-h}^{\eta} \rho g z \, \mathrm{d}z\right) \tag{3.25}$$

If we again time-average over a wave-period (3.1), the LHS becomes

$$\frac{\partial}{\partial t}(\mathrm{KE} + \mathrm{PE}) = \frac{\partial E}{\partial t}$$

where wave kinetic, potential, and total energy (KE, PE, and E) are defined in (3.5), (3.2), and (3.6).

Now, recalling the idea of a flux conservation relationship (3.10), we now have wave energy E and wave energy flux F. Combining the LHS and RHS of the depth-integrated (3.23) we get for linear waves,

$$\frac{\partial E}{\partial t} + \nabla \cdot (E\mathbf{c}_g) = 0, \tag{3.26}$$

which looks like a version of the 1D wave equation. This equation is valid unless wave energy is created (by wind generation) or destroyed (by wave breaking or bottom friction). It also assumes that there are no currents that could refract the wave field.

The statement (3.26) can be more generalized as a *wave-action* conservation equation. Such an equation can apply to a variety of linear wave situations from surface gravity waves, to internal waves, to sound waves. This is a topic that will be addressed in later parts of the course when we focus on inhomogeneous media. But keep (3.26) in mind as it will appear in various guises later on.

3.4 Extension to Real Surface Gravity Waves

1. Spectra describes the variance distribution of η in frequency and direction: $S_{\eta\eta}(f, \theta)$. Also could be in wave number component: $S_{\eta\eta}(k_x, k_y)$ via the dispersion relationship. A fixed instruments measures in f direction. Satellite might measure in $\mathbf{k}$ directly.

2. Curiously wave energy is linearly related to variance (3.8). So for real waves wave energy can be written as a function of f and θ

$$E(f, \theta) = \rho g S_{\eta\eta}(f, \theta)$$

3. Now energy conservation becomes

$$\frac{\partial E(f,\theta)}{\partial t} + \nabla \cdot [(E(f,\theta)c_g(f)] = 0,$$

4. Can add sources and sinks to this:

$$\frac{\partial E(f,\theta)}{\partial t} + \nabla \cdot (E(f,\theta)c_g(f) = S_{\text{wind}} - S_{\text{breaking}} + S_{nl}$$

5. How well does this work? Wave forecasting

3.5 Problem Set

1. Confirm for yourself that the units of (3.26) work out. What are the units of Ec_g?

2. Take a look at the CDIP wave model output. From google earth, figure out what the approximate distance is from Harvest platform to San Diego. Assume deep-water $kh \gg 1$. If a $T = 18$ s swell arrives at Harvest at midnight, how long does it take for the swell to arrive at San Diego (assume no islands). How long for $T = 8$ s waves?

3. Assume linear monochromatic waves with amplitude a and frequency f are propagating in the $+x$ direction on bathymetry that varies only in x, *i.e.,* $h = h(x)$. If the waves field is steady, and there is no wind-wave generation or breaking, then (3.26) reduces to
$$\frac{d}{dx}(Ec_g) = 0. \tag{3.27}$$
Assuming that the dispersion relation and energy conservation hold for variable water depth,

   - In deep water, what is the wave height H dependence on water depth h?
   - In shallow-water, what is the wave height H dependence on water depth h?

   In both cases one can derive a scaling for $H \sim f(h)$.

4. Consider the equation for $\phi(x,t)$ that obeys the equation
$$\phi_t + \phi_{xxx} = 0, \qquad -\infty < x < +\infty \tag{3.28}$$
on the infinite domain. In the Chapter 1 problem set, the dispersion relation $\omega = \omega(k)$ for (3.28) was derived.

   (a) By multiplying (3.28) by ϕ, integrate by parts and time-average over the plane wave solution $\phi = a\cos(kx - \omega(k)t)$ to form an energy equation of the form
$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0$$

   (b) Write the average energy flux F as a velocity times energy. How does this velocity relate to the dispersion relationship and phase velocity? Does phase and energy always move in the same direction? Does phase or energy move faster?

5. Consider now the physical variable $\phi(x,t)$ obeys the equation
$$\phi_{tt} - \phi_{xx} + \phi = 0, \qquad -\infty < x < +\infty$$
on the infinite domain. First, derive the dispersion relationship. The, repeat (a) and (b) for the above question. However, for (a), multiply by ϕ_t and integrate by parts.

Chapter 4

Linear Surface Gravity Waves C, Dispersion, Group Velocity, and Energy Propagation

4.1 Group Velocity

In many aspects of wave evolution, the concept of "group velocity" c_g plays a central role. We've seen it as the velocity associated with the wave energy flux and thus it is known as the "speed of energy propagation", but it is actually more. For example, if a packet of dispersive waves are created suddenly near $x = 0$ at time $t = 0$, the place to look for waves of wavenumber k at time t is not at $x = ct$ but rather at $x = c_g(k)t$

$$c_g = \frac{\partial \omega}{\partial k}. \tag{4.1}$$

This is called the stationary phase problem. As we saw in the last chapter, this *group velocity* c_g is associated with the speed of energy propagation. It is also associated with wavenumber k propagation. Weird. We therefore begin by exploring group velocity in a simple example, a classic stationary phase problem, and then looking at conservation equations for k and ω.

4.2 Slowly Varying Wavetrain, take one: Two-waves

The intrinsic nature of "group velocity" c_g is exhibited in its simplest form by a flow made up of precisely two equal-amplitude plane waves. We choose deep water (*i.e.,* short) gravity waves in one space dimension for our example. The dispersion relation is

$$\omega^2 = gk$$

A two-wave solution is

$$\eta = A\left[\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)\right] \tag{4.2}$$

where $\omega_1 = \sqrt{gk_1}$ and $\omega_2 = \sqrt{gk_2}$. By the cosine angle formula

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right],$$

the two wave case (4.2) can be written as

$$\eta = [2A\cos(\Delta kx - \Delta\omega t)]\cos(\bar{k}x - \bar{\omega}t),$$

where

$$\bar{k} = \frac{k_1+k_2}{2} \qquad \bar{\omega} = \frac{\omega_1+\omega_2}{2},$$
$$\Delta k = \frac{k_1-k_2}{2} \qquad \Delta\omega = \frac{\omega_1-\omega_2}{2}.$$

The interesting case is when $|\Delta k| \ll |\bar{k}|$. Then the factor

$$\mathcal{A}(x,t) = 2A\cos(\Delta kx - \Delta\omega t) \tag{4.3}$$

is interpreted as the *"slowly varying" envelope* of the *carrier* wave $\cos(\bar{k}x - \bar{\omega}t)$ (Figure 4.2). The *envelope* has amplitude $2A$, wavelength Δk, and frequency $\Delta\omega$. *Does one ever see this on the surface of the ocean? What radio band utilizes this principle to transmit audio?.*

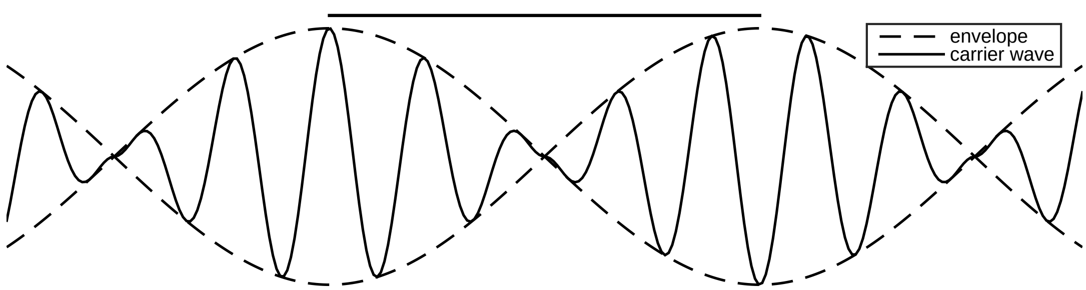


Figure 4.1: Carrier wave and envelope (4.3).

Crests and troughs of the carrier wave move at the phase speed $c = \bar{\omega}/\bar{k}$, but the envelope moves at speed $\Delta\omega/\Delta k$ which as $\Delta k \to 0$ has a limit of

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}.$$

Holy smokes! The propagation speed c_g of the slowly-varying wave envelope (or wave group) is the same as the speed associated with the wave energy flux $F = c_g E$ in the equation (4.23),

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(c_g E) = 0, \tag{4.4}$$

Two different approaches give the same result for c_g. Without all the fancy energy flux calculations of the previous chapter, we could divine that slowly varying wave packets propagate with c_g.

4.3 Slowly varying wavetrain: Take 2 - Deep Water Wave Packet

4.3.1 Setup of Wave Packet Initial Condition

Lets consider deep water waves in one space dimension, but allow for an arbitrary initial condition that has the form of a "packet". Let $\eta(x,t)$ be the free surface elevation, and let the initial condition on η be given as

$$\eta(x,0) = f(x).$$

We will assume that $f(x)$ is nonzero only around $x = 0$, *i.e.,* that $f(x)$ is a packet. What kind of form could this take. A simple form might be

$$f(x) = \exp(-x^2/L_0^2)\cos(k_0 x), \tag{4.5}$$

which is made up of a *carrier* wave $\cos(k_0 x)$ and an envelope $\mathcal{A} = \exp(-x^2/L_0^2)$, where L_0 is the width of the packet. A case that will be interesting is that where the wave packet width is much larger than a wave-length, $L_0 \gg 2\pi k_0^{-1}$ or $L_0 k_0/(2\pi) \gg 1$. See for example the example wave packet in Figure 4.3.1. Here, we investigate how such a packet propagates knowing only the initial condition and the dispersion relationship.

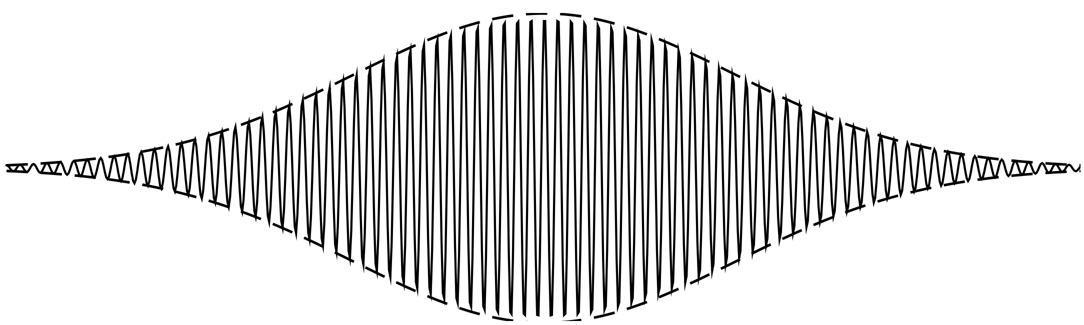


Figure 4.2: Slowly varying wave packet $\exp(-x^2/L_0^2)\cos(k_0 x)$.

Recall that the Fourier representations of η and f are

$$\eta(x,t) = \int_{-\infty}^{\infty} \hat{\eta}(k,t) e^{\mathrm{i}kx}\, \mathrm{d}k,$$
$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{\mathrm{i}kx}\, \mathrm{d}k,$$

where the Fourier transform is

$$\hat{\eta}(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta(x,t) e^{-\mathrm{i}kx}\, \mathrm{d}x,$$

and $\hat{\eta}(k,t) = \hat{\eta}(-k,t)^*$ because $\eta(x,t)$ is real. In our specific example of (4.5) the Fourier transform of $f(x)$ is

$$\hat{f}(k) \propto \exp(-L_0^2(k-k_0)^2), \tag{4.6}$$

which can be found in many books or internet-searches. *Question: Can you derive this?*

4.3.2 Full Solution

With the initial condition of a packet $f(x)$ and its Fourier transform $(\hat{f})$, we can write a solution for the evolution of the entire wave field $\eta(x,t)$ with all its wave components following the analysis of the wave equation in Chapter 1. In analogy with the solution for the wave equation, the general solution for deep water waves is

$$\eta(x,t) = \int_{-\infty}^{\infty} \left[\hat{a}(k)e^{\mathrm{i}(kx-\omega(k)t)} + \hat{b}(k)e^{\mathrm{i}(kx+\omega(k)t)}\right] \mathrm{d}k, \tag{4.7}$$

where $\omega(k) = (gk)^{1/2}$ and

$$\hat{\eta}(k,t) = \left[\hat{a}(k)e^{-\mathrm{i}\omega(k)t} + \hat{b}(k)e^{\mathrm{i}\omega(k)t}\right].$$

The coefficients $\hat{a}(k)$ and $\hat{b}(k)$ are chosen to meet the initial conditions. From the initial condition $\eta(x,0) = f(x)$ we obtain

$$\hat{a}(k) + \hat{b}(k) = \hat{f}(k).$$

As in Chapter 1, to determine both $\hat{a}(k)$ and $\hat{b}(k)$ we obviously need an additional initial condition as the complete initial conditions for any mechanical system are the *position* and *velocity* of particles.

We take, for our additional initial condition, that the surface be initially at rest,

$$\frac{\partial \eta}{\partial t} = 0.$$

As with the wave equation (Chapter 1), this implies

$$-\mathrm{i}\omega\hat{a}(k) + \mathrm{i}\omega\hat{b}(k) = 0,$$

so that

$$\hat{a}(k) = \hat{b}(k) = \frac{\hat{f}(k)}{2}.$$

Thus the exact solution to our problem is

$$\eta(x,t) = \int_{-\infty}^{\infty} \frac{\hat{f}(k)}{2}\left[e^{\mathrm{i}(kx-\omega(k)t)} + e^{\mathrm{i}(kx+\omega(k)t)}\right] \mathrm{d}k, \tag{4.8}$$

which could be integrated numerically or otherwise if $\hat{f}(k)$ is known. However, generally the form (4.8) is not convenient.

Now (4.8) has two very similar terms just propagating in different directions but otherwise treated in precisely the same way. To avoid unnecessary text, we will temporarily pretend that only one term is present, *i.e.*,

$$\eta(x,t) = \int_{-\infty}^{\infty} \hat{f}(k) e^{\mathrm{i}(kx-\omega(k)t)}. \tag{4.9}$$

The combination of both $+x$ and $-x$ solutions is left to the reader.

4.3.3 Approximate Solution for slowly varying packet

Now consider the case of a slowly varying packed with initial condition $f(x)$ given by (4.5) and its Fourier transform (4.6)

$$\hat{f}(k) \propto \exp(-L_0^2(k-k_0)^2),$$

where $k_0 L_0 \gg 1$. This means that the packet is broad and slowly varying (see Figure 4.3.1) or that its Fourier transform $\hat{f}(k)$ is *narrow* around a peak at $k = k_0$. With this particular initial condition, we can evaluate the integral (4.8) by Taylor series expanding the dispersion relationship around $k = k_0$, *i.e.*,

$$\omega(k) = \omega_0(k_0) + k'\omega'(k_0) + \underbrace{\frac{1}{2}k'^2\omega''(k_0) + \ldots}_{\text{neglected}}$$

where $k' = k - k_0$ and $\omega' = \partial\omega/\partial k = c_g$. Next we will only consider terms propagating to the right. This becomes

$$\begin{aligned}
\eta(x,t) &= \frac{1}{2}\int_{-\infty}^{\infty} \hat{f}(k')[e^{\mathrm{i}(k_0x-\omega_0t)} + e^{\mathrm{i}k'(x-c_gt)} + \ldots]\,\mathrm{d}k' \\
&\approx \frac{1}{2}e^{\mathrm{i}(k_0x-\omega_0t)}\int_{-\infty}^{\infty} \hat{f}(k')e^{\mathrm{i}k'(x-c_gt)}\,\mathrm{d}k' \\
&\approx \frac{1}{2}e^{\mathrm{i}(k_0x-\omega_0t)}f(x-c_gt)
\end{aligned}$$

where the last term comes from the properties of Fourier transforms. Which says that the solution is a fast carrier wave of the form $\cos(k_0x-\omega_0t)$ and then an envelope that propagates with the group velocity, $f(x-c_gt)$. What is interesting is that because $c_g = c/2$, the fast carrier waves propagate through the envelope just as seen in the film "Waves Across the Pacific"

4.3.4 Higher Order Solution: What happens next?

At the start of the subsection we mentioned the initial condition (4.5) of a wave packet of width L_0. Can we say anything about the width of the packet as time-evolves? Does it change? Stay the same? Yes one can say something about the packet size evolution. Imagine first that the system is non-dispersive so that ω'' and all higher derivatives are equal to zero. How would the packet width evolve? Well, it would remain constant. Why? Now we neglected higher order terms in the Taylor series expansion of the dispersion relationship, such as those of the form $(1/2)k'^2\omega''(k_0)$. What if $\omega''(k_0)$ is not zero? *Question: What would be the effect of this qualitatively on the wave packet? Would it widen or narrow? For extra credit see the problem set...*

4.4 A confined disturbance at long time: Stationary phase

Previously, we examined the velocity of the envelop of the wave packet whose envelope varied slowly relative to the carrier wave (k_0L_0 was large) and showed that the envelope propagated with $c_g = \mathrm{d}\omega/\mathrm{d}k$. In this section, instead we seek to examine the *long time* evolution of an arbitrary but confined disturbance. This is a classic problem called *stationary phase.* It has application to how wave prediction was performed prior to modern wave models. The basic setup is this. What if there is a big disturbance far away. At what time will we see waves of a particular frequency and how big will those waves be? How will the wave height change even farther away? The method of stationary phase helps provide insight into these questions.

To begin, we seek an approximation to (4.8) that is valid for large x and t based upon the assumption that $f(x) \neq 0$ only within a finite distance of $x = 0$. But really any initial disturbance $f(x)$ that is contained or has $f(x) = 0$ for some $|x|$ will work. For example, this could be a storm in the Southern Ocean. Or an earthquake radiating seismic energy in the geophysics world. By requiring that $f(x)$ be confined to some region near $x = 0$, this forces the Fourier transform $\hat{f}(k)$ to be smooth. This is in contrast to the slowly varying wave packet of Section 4.3 where we assumed that $\hat{f}(k)$ was zero in regions where the dispersion relationship varied significantly. This means that this method can have a broader range of initial disturbances provided they are confined, but is limited to only long times (and long distances).

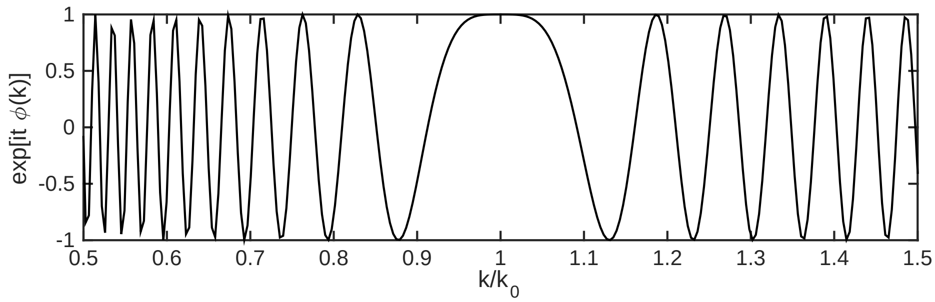


Figure 4.3: Example evaluation of $\exp[\mathrm{i}t\phi(k)]$ near k_0 for the stationary phase approximation.

To examine the large x and t behavior of (4.9), we rewrite (4.9) in the form

$$\eta(x,t) = \int_{-\infty}^{\infty} \hat{f}(k) \exp\left[\mathrm{i}t\phi(k)\right]\,\mathrm{d}k, \tag{4.10}$$

where

$$\phi(k) = \frac{kx}{t} - \omega(k). \tag{4.11}$$

Now, let $t \to \infty$ with the ratio x/t fixed (but arbitrary). This obviously means that x gets really big as well. Within the integral (4.10) for very large, even small changes in $\phi(k)$ with k will cause rapid oscillations in $\exp[\mathrm{i}t\phi]$ (Figure 4.3). Thus, with $\hat{f}(k)$ being smooth, these oscillations will produce canceling contributions to the integral (4.10). This is true except where $\phi(k)$ is constant in k or where $\mathrm{d}\phi/\mathrm{d}k = \phi'(k) = 0$, because if $\phi'(k) = 0$ then changes in k produce no change in $\phi(k)$. This is the concept of *stationary phase* - that is the phase ϕ is relatively constant. With this concept, for a smooth $\hat{f}(k)$ as $t \to \infty$, the dominant contributions to the integral (4.10) come from all the wavenumbers k_0 (Figure 4.3) at which

$$\frac{\mathrm{d}\phi(k_0)}{\mathrm{d}k} = \frac{x}{t} - \frac{\mathrm{d}\omega(k_0)}{\mathrm{d}k} = 0. \tag{4.12}$$

This means that for large t, the location of the k_0 component of the wave field (which has many components) will be at $x = c_g(k_0)t$. That is, at this $x = c_g(k_0)t$ at large time we will locally see a wave with wavenumber k_0. This is consistent with what we knew before with slowly varying wave trains, but is true for arbitrary disturbances at long time as well.

Therefore, we approximate (4.10) at large t and fixed x/t as

$$\eta(x = c_g(k_0)t, t) \approx \sum_{k_0} \int_{k_0-\Delta k}^{k_0+\Delta k} \hat{f}(k_0) e^{\mathrm{i}t\phi(k)}\,\mathrm{d}k, \tag{4.13}$$

where the summation is over all k_0 that satisfy (4.12). For the deep water dispersion relationship ($\omega^2 = gk$), there will only be one k_0 but for other wave systems and corresponding dispersion relationships, there could be more than one!

How big is the amplitude of this wave? Now, near k_0, we Taylor series expand around $(k-k_0)$ resulting in

$$\phi(k) = \phi(k_0) + \phi'(k_0)(k-k_0) + \frac{1}{2}\phi''(k_0)(k-k_0)^2 + \ldots \tag{4.14}$$

$$\approx \left[\frac{k_0 x}{t} - \omega(k_0)\right] + 0 - \frac{1}{2}\omega''(k_0)(k-k_0)^2. \tag{4.15}$$

Here we assume that $\omega''(k_0) \neq 0$ and stop after that term. (*The case where* $\omega''(k_0) = 0$ *is way more algebra and will not be considered here.*) Note that here we've gone to a higher order expansion including $\omega''(k_0)$ whereas in Section 4.3 we only went to ω'. With this expansion (4.14), the expression (4.13) simplifies to,

$$\eta(x = c_g(k_0)t, t) \approx \sum_{k_0} \hat{f}(k_0) e^{\mathrm{i}(k_0 x - \omega(k_0)t)} \int_{k_0-\Delta k}^{k_0+\Delta k} \exp\left[-\frac{\mathrm{i}t}{2}\omega''(k_0)(k-k_0)^2\right] \mathrm{d}k. \tag{4.16}$$

This is starting to look familiar. There is a carrier wave $\exp[\mathrm{i}(k_0 x - \omega(k_0)t)]$, an amplitude $\hat{f}(k_0)$, and then an integral that depends on the dispersion relationship.

To evaluate the integral in (4.16), change the integration variable to

$$\alpha = (k-k_0)\sqrt{\frac{t}{2}|\omega''(k_0)|},$$

and the integral in (4.16) becomes

$$\frac{1}{\sqrt{\frac{t}{2}|\omega''(k_0)|}} \int_{-\Delta\alpha}^{+\Delta\alpha} \exp\left[-\mathrm{i}\alpha^2 \operatorname{sgn}\left(\omega''(k_0)\right)\right] \mathrm{d}\alpha, \tag{4.17}$$

where

$$\mathrm{d}\alpha = \mathrm{d}k\sqrt{\frac{t}{2}|\omega''(k_0)|}.$$

As $t \to \infty$, $\mathrm{d}\alpha \to \infty$ for any $\mathrm{d}k$, and the integration limits in (4.17) may be replaced by $(-\infty, \infty)$. This is very satisfying, because Δk is an arbitrary quantity. It only remains to evaluate the integral (see Appendix),

$$\int_{-\infty}^{\infty} e^{-\mathrm{i}\alpha^2 \operatorname{sgn}(\omega''(k_0))} \mathrm{d}\alpha = \sqrt{\pi} \exp\left(\frac{-\mathrm{i}\pi}{4} \operatorname{sgn}[\omega''(k_0)]\right). \tag{4.18}$$

Combining all results, for large t near $x = c_g(k_0)t$,

$$\eta(x = c_g(k_0)t, t) \approx \sum_{k_0} A(t; k_0) e^{\mathrm{i}(k_0 x - \omega(k_0)t)} e^{-\mathrm{i}\pi \operatorname{sgn} \omega''(k_0)/4} \tag{4.19}$$

where the amplitude A is given by

$$A(t; k_0) = \frac{\sqrt{\pi}\hat{f}(k_0)}{\sqrt{\frac{t}{2}|\omega''(k_0)|}}. \tag{4.20}$$

The approximation method used to get (4.19) and (4.20) is called the method of *stationary phase*. This version presented is somewhat less rigorous than that given by Lighthill, who points out that the procedure can also be done as a special case of the method of *steepest descents*.

4.4.1 Discussion Points

Going back to our full solution (4.8), we easily see that the second term, whose analysis we deferred, makes no contribution on $x \gg 0$. For $x \ll 0$, the first term is negligible, and the second term makes a contribution analogous to the one we have worked out.

Where has the assumption that $f(x) = 0$ away from $x = 0$ actually been used? In the assumption that $\hat{f}(k)$ is smooth! For if $\hat{f}(k)$ has "*infinitely* sharp corners", then even the rapid oscillations of $e^{it\phi}$ may not cause cancellation. The fact that $f(x) = 0$ for $|x|$ sufficiently large forces $\hat{f}(k)$ to be smooth in the sense that rapid changes in any function are associated with "high wavenumber" in its Fourier transform.

We next examine the physical content of (4.19). At large x and t, imagine an observer with a special preference for the wavenumber k_0, who is willing to suffer any inconvenience to always be where the local wavenumber is k_0. This observer must move at the velocity $c_g(k_0)$, and for him k_0 in (4.19) is truly a constant. However, since $c_g(k_0) = (1/2)c(k_0)$ for surface waves, crests and troughs will move rightward relative to the observer, no matter what k_0. What about the amplitude of the waves for this observer? From (4.20), the amplitude decays as $t^{-1/2}$ but the rate of decay - the "diffusivity" if you will (aside: why is that analogy appropriate?) - depends upon the dispersion relation and $\omega''(k_0)$. For smaller $\omega''(k_0)$, the amplitude decay is less that for larger $\omega''(k_0)$.

4.4.2 Development of Energy Conservation Equation

At $x = c_g(k_0)t$, the complex amplitude (4.20) is

$$A(t; k_0) = \frac{\sqrt{\pi}\hat{f}(k_0)}{\sqrt{\frac{t}{2}|\omega''(k_0)|}}.$$

We can now develop a conservation equation for the quantity $Q(t)$,

$$Q(t) = \int_{x_1}^{x_2} |A|^2 \, \mathrm{d}x = 2\pi \int_{x_1}^{x_2} \frac{\hat{f}(k)\hat{f}^*(k)}{t|\omega''(k)|} \, \mathrm{d}x. \tag{4.21}$$

where we assume $\omega''(k) > 0$. Now, if we specify that x_1 and x_2 are traveling at the group velocity, we can use the relationship $x = \omega'(k)t$ and thus $\mathrm{d}x = \omega''(k)t \, \mathrm{d}k$ to transform (4.21) to

$$Q(t) = \int_{x_1}^{x_2} |A|^2 \, \mathrm{d}x = 2\pi \int_{k_1}^{k_2} \hat{f}(k)\hat{f}^*(k) \, \mathrm{d}k, \tag{4.22}$$

where k_1 and k_2 are given by $x_1 = \omega'(k_1)t$, etc. What this means is that if k_1 and k_2 are fixed, then $Q(t)$ (4.22) is a constant! That is the integral of $|A|^2$ (4.21) where the limits move with the appropriate group velocity. This means that the total $|A|^2$ between any pair of points moving with the group velocity is conserved. As the separation between the points grows with t, $|A|^2$ decreases with t^{-1}.

Next, for the case of (4.22), we know that $\mathrm{d}Q/\mathrm{d}t = 0$, but we can also calculate $\mathrm{d}Q/\mathrm{d}t$ based on (4.21) explicity which becomes using Leibniz rule,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{x_1}^{x_2} |A|^2 \, \mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial |A|^2}{\partial t} \, \mathrm{d}x + |A(x_2)|^2 \omega'(k_2) - |A(x_1)|^2 \omega'(k_1) = 0.$$

As $x_2 - x_1 \to 0$, we arrive at a conservation equation for $|A|^2$ of

$$\frac{\partial |A|^2}{\partial t} + \frac{\partial (c_g |A|^2)}{\partial x} = 0.$$

This looks curiously similar to our Energy Conservation equation derived before where $E \propto |A|^2$. This derivation works for all wave-systems and dispersion relationships but at long-times and for $\omega'' \neq 0$. Why is long-time essential here? It means that enough time has passed such that the arbitrary initial disturbance is now slowly varying. The essence of this derivation and discussion follows that of Whitham. It is very worthwhile to read Chapter 11.1-11.6 of Whitham.

Appendix: Evaluation of Integral

Some care is required to evaluate (4.18),

$$\int_{-\infty}^{\infty} e^{-\mathrm{i}\alpha^2 \,\mathrm{sgn}(\omega''(k_0))} \, \mathrm{d}\alpha,$$

because this integral exists only if carefully interpreted. Suppose $\omega''(k_0) > 0$ and replace (4.18) by

$$\int_{-\infty}^{\infty} e^{-(\mathrm{i}+\epsilon)\alpha^2} \, \mathrm{d}\alpha$$

where $\epsilon > 0$ is real. Now there is no problem with convergence since

$$\left| e^{-(\mathrm{i}+\epsilon)\alpha^2} \right| = e^{-\epsilon\alpha^2} \to 0 \quad \text{as} \quad \alpha \to \pm\infty$$

Then

$$\int_{-\infty}^{\infty} e^{-(\mathrm{i}+\epsilon)\alpha^2} \, \mathrm{d}\alpha = \frac{\sqrt{\pi}}{\sqrt{\mathrm{i}+\epsilon}} \to \sqrt{\pi} e^{-\mathrm{i}\frac{\pi}{4}} \quad \text{as} \quad \epsilon \to 0$$

where we select the branch of $\sqrt{\mathrm{i}+\epsilon}$ with positive real part. Treating the case $\omega''(k_0) < 0$ similarly, we find that

$$\int_{-\infty}^{\infty} e^{-\mathrm{i}\alpha^2 \operatorname{sgn}(\omega''(k_0))} = \sqrt{\pi} \exp\left(\frac{-\mathrm{i}\pi}{4} \operatorname{sgn} \omega''(k_0)\right).$$

Problem Set

1. Consider the 1-9 August storm discussed in Snodgrass et al. (1966), the paper associated with the movie "Waves Across the Pacific". In particular see Sections 5a and 5b. Read enough of the previous sections that you get a sense of what is going on.

   (a) In Figure 16, why do the ridges for the 1-9 August storm slope over the farther away from the storm?

   (b) In Figure 21, top panel: Consider the top panel as representing $|A|^2$. Can you rationalize the attenuation at $f = 0.065$ Hz (65 mc/s), as being from dispersive attenuation from stationary phase? Find the distances between the given locations on the map and use the deep water dispersion relationship.

2. (from Kundu et al., 7.11) The effect of viscosity on the energy of linear deep-water surface waves can be determined from the wave motion's velocity components and the viscous dissipation.

   (a) For incompressible flow, the viscous dissipation of energy per unit mass of fluid is $\epsilon = 2\nu S_{ij}^2$, where S_{ij} is the strain-rate tensor and ν is the fluid kinematic viscosity. Calculate ϵ from the wave solutions in Lecture 2 for deep water waves.

   (b) Vertically integrate the wave ϵ to calculate

   $$D_w = \int_{-\infty}^{0} \epsilon(z) dz$$

   Now a wave energy conservation equation that included viscous effects would look like

   $$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(E c_g) = -D_w \tag{4.23}$$

   where D_w is a function of the wave energy and other wave properties (period?).

   (c) Assuming $\partial_t = 0$, that is a balance between flux divergence and dissipation, write a closed form solution for wave energy or wave amplitude as a function of x (again in deep water).

   (d) Using $\nu = 10^{-6}\ \mathrm{m}^2\,\mathrm{s}^{-1}$, determine the distance necessary for a 50% amplitude reduction for waves with period $T = 2$ s and $T = 20$ s.

   (e) Thinking about "Waves across the Pacific": find the distance from New Zealand to Southern California (or Tutuila to Yakutat in regards to question 1). How much loss of wave energy (or amplitude) would you expect for the waves of frequency $f = 0.065$ Hz?

3. (EXTRA CREDIT) Let

$$\eta(x,t) = \int_{-\infty}^{\infty} \hat{f}(k) e^{+i[kx-\omega(k)t]}\, \mathrm{d}k$$

(a) For the initiation condition (4.5) also shown in Figure 4.3.1,

$$f(x) = \mathrm{Re}\left\{ e^{-x^2/L_0^2} e^{ik_0x} \right\},$$

determine the $\hat{f}(k)$ explicitly (and without approximation). Note the approximate answer is given in (4.6). Regard L_0 and k_0 as constants.

(b) Consider now that $k_0L_0 \gg 1$ so that the initial condition is a "wave packet" of length L_0, and $\omega = \omega(k)$ as a general (unspecified) dispersion relationship. Demonstrate that the solution $\eta(x,t)$ at time t is a wavepacket of length

$$L = L_0\sqrt{1 + \frac{4(\omega''(k_0))^2t^2}{L_0^4}}$$

Do not assume x or t is large.

(c) Can you give a short explanation that establishes an analogous result qualitatively?

Chapter 5

Linear Surface Gravity Waves D, Ray Theory

Up to now, we have only been considering surface gravity wave problems in constant depth. Because the depth is constant the medium in which the waves are propagating is *homogeneous*. In general, this means that the non-wavenumber parameters of the dispersion relationship are fixed. For example, in surface gravity waves

$$\omega^2 = gk \tanh(kh), \tag{5.1}$$

and thus for homogeneous media g and h are constant in space and time. The same applies for any other dispersion relationship such as those for internal waves and sound waves.

Of course, no wave propagates in truly homogeneous media, and in general inhomogeneous media "waves problems" fall into one of two categories. In the first category, the wavelength is *large* compared to the scale of the inhomogeneities in the medium. This category includes typical *scattering* and *diffraction* problems, as well as generation by *localized sources*. Although there are numerous simple examples, these problems are in general quite complicated and require methods which are considerably more sophisticated than the relatively simple plane-wave Fourier analysis we have been using.

In the second category the wavelength is *small* compared to the scale for variations in the medium. The "slow" variations in the medium cause slow variations in the wavenumber and amplitude of the waves. These slow variations of the medium are subject of this chapter. The theory of "slowly varying waves" is relatively simple, and takes the form of a generalized theory of plane waves which we call *ray theory*. Ray theory has two essential parts:

1. *Ray equations*, which describe the propagation, in space and time, of the slowly varying wavenumber (i.e. describe *refraction*)

2. An "energy equation", which describes the propagation of "wave energy" in space and

time. We have already seen energy equations for homogeneous media, and it also is applied to slowly varying inhomogeneous media.

The ray equations can be derived from the simple postulate that the slowly-varying wavenumber and frequency obey the *same* dispersion relationship as do plane waves in an *exactly homogeneous* medium. Interestingly, the ray equations are analogous to Hamilton's equations in mechanics (see the Appendix, Chapter 20),

Ray theory can be considered as the first term in a formal asymptotic expansion in which the "small parameter" is the ratio of the wavelength to the lengthscale of the medium. This asymptotic analysis is often done with *WBK theory*. (For a general introduction see Bender & Orszag, *Advanced Mathematical Methods*, chapter 10.) However, the full WKB theory is usually not required, and we will not use it here, but it is useful in sorting out paradoxes that can develop at the ray theory level of approximation.

5.1 Slowly varying wavetrain: Take 4, Homogeneous Media

Here we start with the assumption of a slowly varying wavetrain and *postulating* that the plane wave dispersion relation is satisfied locally. We will look at a slowly varying plane-wavetrain in homogeneous media. This is another way of looking at the results of Chapter 4. Here we explicity define what is meant by slowly varying. By "slowly varying" we mean that the wavenumber and frequency change only gradually over many oscillations. These slow variations can come about either because the slowly-varying initial condition and dispersion (as in the examples in Chapter 4), or because the medium itself exhibits a slow variation (in fluid depth, for example - to be considered in the next section.

We suppose that the general wave in a slowly-varying medium can be written in the form

$$\eta(x,t) = A(x,t)e^{\mathrm{i}\theta(x,t)} \tag{5.2}$$

where A is the wave envelope and θ is the phase. Now define the *local* wavenumber and frequency, respectively, as

$$k \equiv \frac{\partial \theta}{\partial x} \tag{5.3}$$

$$\omega \equiv -\frac{\partial \theta}{\partial t}. \tag{5.4}$$

Note that, in the special case of a plane wave, $\theta = kx - \omega t$ and k, ω are constants. Directly from the definitions (5.3-5.4), it follows that

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \tag{5.5}$$

This is often interpreted as a "conservation equation for wave crests". (*Why? What form does this equation hold?*) This can be easily extended to multiple space dimensionsl with $\mathbf{k} = \nabla\theta$ or $k_i = \partial\theta/\partial x_i$. However, here we keep things in one-dimension to simplify.

More generally, the wave (5.2) is "locally sinusoidal" if the local wavenumber k varies slowly over a wavelength ($\lambda = 2\pi k^{-1}$). The length-scale of k variation (L_k) will be given by $\partial k/\partial x \approx (k/L_k)$ or $L_k = (k^{-1}\partial k/\partial x)^{-1}$. Thus, for the medium to be slowly varying, the ratio of the wavelenght $\lambda = 2\pi/k$ to L_k must be small or

$$\frac{\lambda}{L_k} \ll 1,$$
$$\frac{\lambda}{L_k} = \frac{1}{kL_k} = \frac{1}{k^2}\frac{\partial k}{\partial x} \ll 1.$$

Similar approach applies to wave frequency resulting in the requirement that

$$\frac{1}{\omega^2}\frac{\partial \omega}{\partial t} \ll 1.$$

The same argument can be made for the amplitude of the wave envelope $A(x,t)$. Spatially $A(x,t)$ must vary slowly so that over a wavelength, the wave envelope does not seem to vary. Stated mathematically, this implies that,

$$\frac{1}{kA}\frac{\partial A}{\partial x} \ll 1. \tag{5.6}$$

Now if A, k, ω change slowly enough so that conditions are "locally sinusoidal", we expect that, to first order, ω, k satisfy the dispersion relation for plane waves in each locality. We therefore *assume*

$$\omega(x,t) = \Omega\left(k(x,t)\right) \tag{5.7}$$

where

$$\omega = \Omega(k)$$

is the dispersion relation for plane waves in homogeneous media. (*Aside: Why change the notation?*) This assumption is the basis for all results to follow. (We use the notation Ω to emphasize explicit functional dependence.) For surface gravity waves, $\Omega(k) = \sqrt{gk\tanh(kh)}$. By (5.5) and (5.7)

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = \frac{\partial k}{\partial t} + \frac{\mathrm{d}\Omega}{\mathrm{d}k}\frac{\partial k}{\partial x} = 0 \tag{5.8}$$

$$\Longleftrightarrow \qquad \left(\frac{\partial}{\partial t} + c_g\frac{\partial}{\partial x}\right) k = 0 \tag{5.9}$$

This states that an observer moving at the group velocity would always see the same local wavenumber, as shown previously in a variety of special cases also based on slowly-varying principles,

$$\frac{\partial \omega}{\partial t} = \frac{\mathrm{d}\Omega}{\mathrm{d}k}\frac{\partial k}{\partial t} = -\frac{\mathrm{d}\Omega}{\mathrm{d}k}\frac{\partial \omega}{\partial x} \tag{5.10}$$

$$\iff \qquad \left(\frac{\partial}{\partial t} + c_g\frac{\partial}{\partial x}\right)\omega = 0 \tag{5.11}$$

Thus the local frequency also propagates at the group velocity c_g. Again, we see that $c_g = \partial\omega/\partial k$ is the crucial parameter for how many wave properties propagate.

5.2 Inhomogeneous slowly-varying media

Up to know in our discussion of surface gravity waves, the water depth h has remained a constant. That is, any parameters in the dispersion relationship

$$\omega(x,t) = \Omega\left(k(x,t); h, g\right) \tag{5.12}$$

aside from k are spatially and temporally constant. In most cases it makes sense that g is constant but does it really make sense that h is constant? In deep water, $\omega = (gk)^{1/2}$ and so any depth variation is irrelevant. But what about shallow water where $\omega = (gh(x))^{1/2}k$? Or more generally when

$$\omega(x,t) = \Omega\left(k(x,t); h(x), g\right). \tag{5.13}$$

This type of situation here the dispersion relationship parameters vary in space or time is called *inhomogeneous media*. You have already encountered this in your previous studies. Think about your Physics-Optics course where light going from air to glass (*e.g.,* a prism) refracts (or bends) because the index of refraction (speed of light) varies. This is wave propagation across inhomogeneous media.

Now, in general, the spatial or temporal variation in the media must also be slow to apply this "locally sinusoidal" concept. *Is this true for light refracting on a prism?* This means that the same constraint applied to the wave envelope A (5.6) and wavenumber k also applies to the dispersion relationship parameters. For shallow water waves, this means that,

$$\frac{1}{kh}\frac{\partial h}{\partial x} \ll 1.$$

How would this affect the conservation equations for k (5.9)?

Equations (5.9) and (5.11) require modification if the *medium* itself varies (slowly) in space or time. Let the dispersion relation $\omega = \Omega(k; h)$ contain the parameter h which is a property of the medium (like the fluid depth). If $h(x,t)$ slowly varies, then the dispersion relationship has two ways that it can depend on the spatial variable,

$$\omega(x,t) = \Omega\left(k(x,t), h(x,t)\right) \tag{5.14}$$

and (5.9) and (5.11) are replaced by

$$\left(\frac{\partial}{\partial t} + c_g \frac{\partial}{\partial x}\right) k = -\frac{\partial \Omega}{\partial h}\frac{\partial h}{\partial x} \tag{5.15}$$

$$\left(\frac{\partial}{\partial t} + c_g \frac{\partial}{\partial x}\right) \omega = \frac{\partial \Omega}{\partial h}\frac{\partial h}{\partial t} \tag{5.16}$$

Equation (5.15) is equivalent to the two equations, which are called the *Ray Equations.*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial \Omega}{\partial k} \tag{5.17}$$

$$\frac{\mathrm{d}k}{\mathrm{d}t} = -\frac{\partial \Omega}{\partial x} \tag{5.18}$$

where the derivatives must be *carefully* interpreted as

$$\frac{\partial \Omega}{\partial k} \equiv \left.\frac{\partial \Omega}{\partial k}\right|_h \equiv c_g(k,h) \quad \text{and} \quad \frac{\partial \Omega}{\partial x} \equiv \left.\frac{\partial \Omega}{\partial x}\right|_k = \frac{\partial \Omega}{\partial h}\frac{\partial h}{\partial x}$$

It is best to regard the ray equations as equations for the changes in x and k seen by an observer traveling along the ray. The first equation merely states that the observer moves at the group velocity. The second equation describes the change in k that the observer sees. This change in k is caused solely by changes in the medium and is called *refraction.*

Equation (5.16), which can now be written

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \left.\frac{\partial \Omega}{\partial t}\right|_{k,x} \tag{5.19}$$

need not actually be solved. Instead we can use (5.14) to find $\omega(x,t)$, once $k(x,t)$ has been found from (5.18). To find $k(x,t)$ using (5.18), one can imagine a continuous infinity of observers distributed along the x-axis. The observers move as specified by (5.18) and the changes the see map out $k(x,t)$.

Aside on relationship to Hamiltonian Mechanics

It is useful to realize that the ray equations are exact analogues of Hamilton's equations for classical mechanics,

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p} \quad , \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q}$$

The analogy is between

$$\begin{aligned} x &\longleftrightarrow q \\ k &\longleftrightarrow p \\ \Omega(x,k) &\longleftrightarrow H(q,p) \\ \omega &\longleftrightarrow \text{energy} \end{aligned}$$

More on this in the Appendix (Chapter 20).

5.2.1 Wave energy conservation

We have already derived wave energy conservation laws before for homogeneous media. This was done by taking momentum equations, multiplying by $\mathbf{u}$, integrating by parts, and eventually averaging over a wave period. Thus, the wave energy E, in addition to k and ω, are *wave averaged* quantities. For example, E does not explicit fluctuate with the fast time and space scales of ω. The wave energy E can be thought of as related to the slowly varying wave envelope squared $E \propto A^2$. In most wave systems that are slowly varying and some other caveats (no wave current interaction), the wave energy followed a wave energy conservation equation where

$$\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{c}_g E) = 0 \tag{5.20}$$

where the wave energy flux is $\mathbf{F} = \mathbf{c}_g E$.

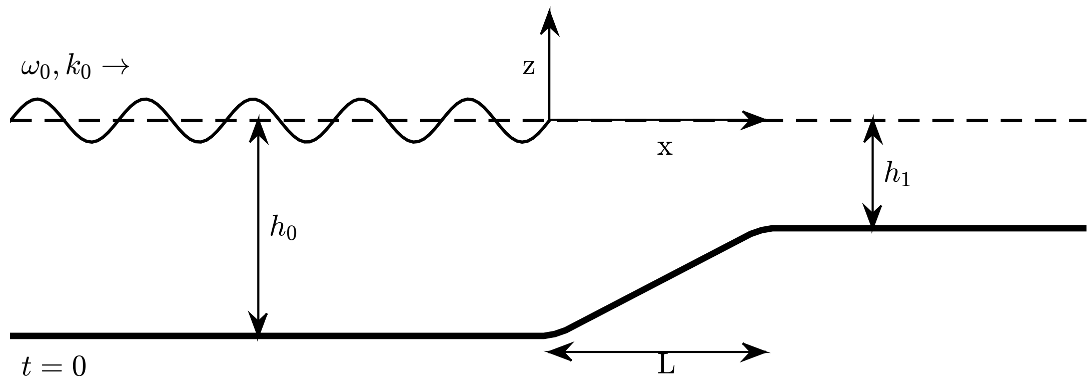
For slowly varying inhomogeneous media (5.20) also applies because locally the wave is sinusoidal and E and $\mathbf{F}$ were locally defined. The spatial derivatives are over the slowly varying spatial scale and not the fast scale associated with k.

There are some situations where ray theory with wave energy conservation break down. These situations can be handled with an asymptotic theory like WKB (see Chapter 17). It turns out, however, that (5.20) *is* valid for inhomogeneous media with no time variations. Time variations in the medium require further generalizations. Rigorous derivations require asymptotic methods like the WKB approximation. The best general heuristic derivation makes use of the concept of a "wave packet". For the moment we accept (5.15–5.20), and proceed to some examples.

5.3 Ray Theory For Shallow Water Waves, Example in one dimension

As an example of the use of the *slowly varying* theory, consider a plane wave approaching a gradual change in depth from a constant-depth region to the left: The wavelength is long compared to the depth but short compared to the width of the region where the depth is varying. Thus slowly varying theory applies. At $t = 0$, the wave-train (of frequency ω_0 and wavenumber k_0) has just reached $x = 0$. The dispersion relationship which holds locally is

$$\omega = (gh(x))^{1/2}k.$$



Thus, the full set of ray equations are,

$$\omega = (gh(x))^{1/2}k \tag{5.21}$$

$$c_g = \frac{\partial \omega}{\partial k} \tag{5.22}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c_g \tag{5.23}$$

$$\left(\frac{\partial}{\partial t} + c_g \frac{\partial}{\partial x}\right) k = -\frac{1}{2}\left(\frac{g}{h}\right)^{1/2} k \frac{\mathrm{d}h}{\mathrm{d}x} \tag{5.24}$$

$$\left(\frac{\partial}{\partial t} + c_g \frac{\partial}{\partial x}\right) \omega = 0 \tag{5.25}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(c_g E) = 0 \tag{5.26}$$

When the wave has passed completely through the transition region, the wave field properties can be deduced as follows: By (5.25), ω is constant everywhere. Thus by (5.21)

$$\sqrt{gh(x)}k(x) = \sqrt{gh_0}k_0$$

or

$$k(x) = \left(\frac{h_0}{h(x)}\right)^{\frac{1}{2}} k_0$$

and the waves shorten as they pass throuh the slope region. This could also be arrived at by (5.24 as well. The energy density is found from (5.26) to be

$$c_g(x)E(x) = c_g(0)E_0 \implies E(x) = \left(\frac{h_0}{h(x)}\right)^{1/2} E_0$$

so that the wave energy increases into shallow water as the energy flux is conserved. As $E \propto a^2$ or H^2 the wave height increases into shallower water. This process is called *shoaling*.

5.3.1 Arrival time

As $\mathrm{d}x/\mathrm{d}t = (gh(x))^{1/2}$, then

$$dt = \frac{\mathrm{d}x}{(gh(x))^{1/2}}$$

and so the leading edge of the wave-train reaches the new depth (after passing through a horizontal distance L) in time

$$t_a = \int_0^L \frac{\mathrm{d}x}{(gh(x))^{1/2}},$$

where t_a is the time a crest arrives at L. Since these waves are nondispersive, t_a is the same for all k.

5.4 Extension of the Ray theory to two (or more) space dimensions

The previous example was just in one direction, but of course ray theory works in two-dimensions (2D) and is much more interesting as it gives rise to wave *refraction*. First we will derive the ray equations in 2D and then give an example.

Again assume

$$\eta(\mathbf{x}, t) = A(\mathbf{x}, t)e^{i\theta(\mathbf{x},t)}$$

where now $\mathbf{x} = (x_1, x_2)$ and *define*

$$k_i \equiv \frac{\partial \theta}{\partial x_i} \quad \text{and} \quad \omega \equiv -\frac{\partial \theta}{\partial t}$$

Again we *suppose* that the plane wave dispersion relationship is locally correct, i.e.

$$\omega(\mathbf{x}, t) = \Omega\left((\mathbf{k}(\mathbf{x}, t), h(\mathbf{x}, t)\right)$$

and find that

$$\frac{\partial k_i}{\partial t} + \frac{\partial \Omega}{\partial k_j}\frac{\partial k_j}{\partial x_i} + \frac{\partial \Omega}{\partial h}\frac{\partial h}{\partial x_i} = 0$$

where the "summation convention" is being used. But

$$\frac{\partial k_j}{\partial x_i} = \frac{\partial k_i}{\partial x_j}$$

by definition *(Question: do you see why?)*. Thus

$$\left(\frac{\partial}{\partial t} + C_{g_j}\frac{\partial}{\partial x_j}\right) k_i = -\frac{\partial \Omega}{\partial h}\frac{\partial h}{\partial x_i} \tag{5.27}$$

and similarly

$$\left(\frac{\partial}{\partial t} + C_{g_j}\frac{\partial}{\partial x_j}\right) \omega = +\frac{\partial \Omega}{\partial h}\frac{\partial h}{\partial t} \tag{5.28}$$

The energy equations becomes

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}(C_{g_j} E) = 0 \tag{5.29}$$

$$\tag{5.30}$$

provided $\partial h/\partial t = 0$. However, in many cases the medium does not vary in time. Equations (5.27–5.28) may best be solved as follows: Let x_i be the location of an observer moving at the group velocity. Then

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial \Omega}{\partial k_i} \tag{5.31}$$

and

$$\frac{\mathrm{d}k_i}{\mathrm{d}t} = -\frac{\mathrm{d}\Omega}{\mathrm{d}h}\frac{\mathrm{d}h}{\mathrm{d}x_i} \tag{5.32}$$

where

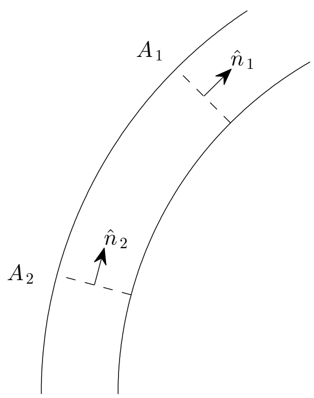
$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{C}_g \cdot \nabla.$$

Equations (5.31–5.32) are a complete set of 6 equations in 6 unknowns $\mathbf{k}, \mathbf{x}$ which may be solved given an initial wavenumber $\mathbf{k}$ and location $\mathbf{x}$. (There is no need to use (5.28) except as a check, although it can be substituted for one equation of (5.31–5.32).) The resulting trajectory in $\mathbf{x}$-space is called a *ray*. If the medium is homogeneous, then by (5.32), $\mathbf{k}$ is constant along the ray.

The most common application of the ray-tracing equations (5.31-5.32) is to time-invariant media in which waves of some fixed frequency have reached a steady state $\left(\frac{\partial k_i}{\partial t} = \frac{\partial E}{\partial t} = 0\right)$. The raws are traced using (5.31) and (5.32), then the energy equation is applied. The energy equation (5.29) in steady state has the form

$$\nabla \cdot (\mathbf{C}_g E) = 0 \tag{6}$$

which can be solved by the "ray tube" method. A ray tube is defined as a volume whose sides are tangent to $\mathbf{C}_g$.



Integration of (6) over a slice of the ray tube gives

$$\int\int_{A_1} E_1 \mathbf{C}_{g_1} \cdot \hat{n}_1 \, \mathrm{d}A_1 = \int\int_{A_2} E_2 \mathbf{C}_{g_2} \cdot \hat{n}_2 \, \mathrm{d}A_2$$

which approximates to

$$E_1 \mathbf{C}_{g_1} \cdot \hat{n}_1 A_1 = E_2 \mathbf{C}_{g_2} \cdot \hat{n}_2 A_2$$

for a thin enough tube. What happens if rays converge or diverge? Then the areas (A_1 and A_2) change which has implications for the wave energy.

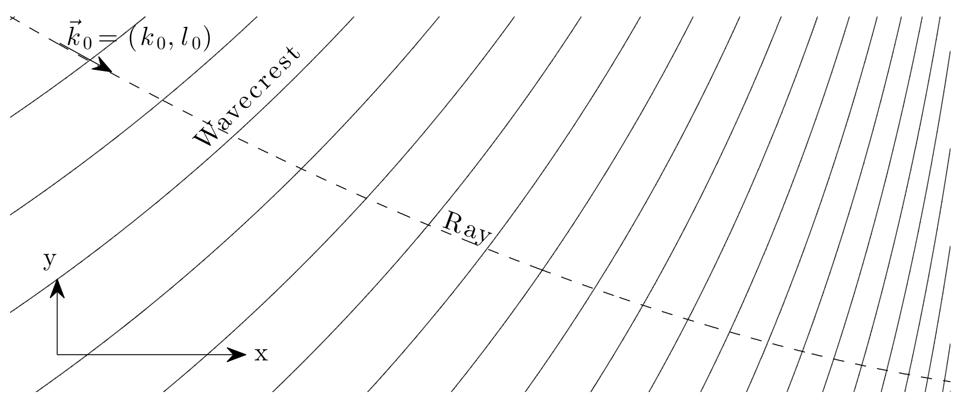
5.5 Ray Tracing of Shallow Water Waves Example in two dimensions:

The dispersion relationship is now

$$\omega^2 = (gh(x))(k^2 + l^2)$$

where (k, l) are the (x, y) components of wavenumber, respectively.

5.5.1 Obliquely incident ray moving into shallower water



Reconsider the previous example of a shoaling gravity wave but now for the case of an obliquely incident plane wave. with $\mathbf{k} = (k, l)$ the wavevector, the ray tracing equations become

$$\omega = \sqrt{gh(x)(k^2 + l^2)} = \Omega(\mathbf{k}, h) \qquad (5.33)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial \Omega}{\partial k} = c_{gx} \qquad (5.34)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial \Omega}{\partial l} = c_{gy} \qquad (5.35)$$

$$\frac{\mathrm{d}k}{\mathrm{d}t} = -\frac{\partial \Omega}{\partial h}\frac{\partial h}{\partial x} \qquad (5.36)$$

$$\frac{\mathrm{d}l}{\mathrm{d}t} = 0 \qquad (5.37)$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = 0 \qquad (5.38)$$

By (5.38), ω is constant along each ray. Thus,

$$gh_0(k_0^2 + l_0^2) = gh(x)(k^2 + l^2) \tag{5.39}$$

and by (5.37) implies that the alongshore wavenumber l is also conserved

$$l(x) = l_0.$$

Thus (5.39) implies

$$k^2(x) = \frac{h_0}{h(x)}(k_0^2 + l_0^2) - l_0^2 \tag{5.40}$$

The slope of the rays $\mathrm{d}y/\mathrm{d}x$ are given by (5.34) and (5.35)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\partial\Omega}{\partial l}}{\frac{\partial\Omega}{\partial k}} = \frac{l}{k} \tag{5.41}$$

where the last result $\mathrm{d}y/\mathrm{d}x = l/k$ is true only for non-dispersive systems. This non-dispersive result says that rays are tangent to the local wavenumber $\mathbf{k}$. Equation (5.41) implies

$$\sin^2\theta = \frac{(dy)^2}{(dx)^2 + (dy)^2} = \frac{l^2}{l^2 + k^2} = \frac{ghl^2}{\omega^2} = \frac{ghl_0^2}{gh_0(k_0^2 + l_0^2)} \tag{5.42}$$

This implies that with phase velocity $c(x) = \sqrt{gh(x)}$

$$\frac{\sin\theta(x)}{c(x)} = \text{constant} \tag{5.43}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin\theta(x)}{c(x)}\right) = 0 \tag{5.44}$$

which is the continuous version of Snell's law. Thus, if $h \to 0$ then $c \to 0$ and $\sin\theta \to 0$, and the rays become parallel to the x-axis.

It is useful to compare the continuous version of Snell's law (5.43) with that for discontinuous media such as light passing from air to glass or sound from air to water (recall the acoustics chapter 6 problem set). As you have already derived, for two media with phase speeds c_1 and c_2, the statement of Snell's law is

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2} \tag{5.45}$$

which is essentially identical to (5.43). This is one reason why ray theory has it's name and why it is also called *Geometric Optics* in analogy to light propagation. In fact, light propagation through continuously varying media will also follow this continuous ray theory.

5.5.2 Obliquely incident ray moving into deeper water: Caustics

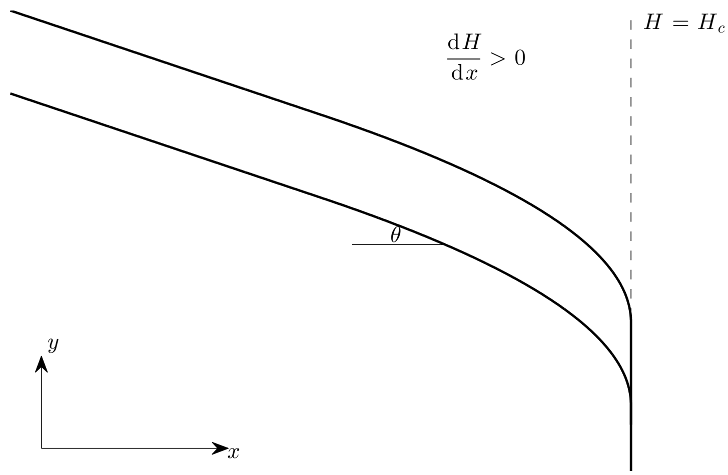
If $\mathrm{d}h/\mathrm{d}x > 0$ the situation is different. Recall from (5.40) that

$$k^2(x) = \frac{h_0}{h(x)}(k_0^2 + l_0^2) - l_0^2,$$

and as $l_0^2 > 0$ and $h(x) > h_0$ then eventually at some depth h_c the cross-shore wavenumber $k = 0$, implying that the waves are propagating alongshore. This critical depth h_c can be found by setting $k = 0$ in (5.40) and solving for the appropriate depth,

$$h_c = \frac{(k_0^2 + l_0^2)}{l_0^2} h_0,$$

also where $(\sin\theta = 1)$ (or $\theta = \pi/2$) and the rays become parallel to the y-axis.



Since $k < 0$ for $h > h_0$ the waves evidently do not pass the line $h = h_c$, which is called a *caustic*. Note that $h_c = \infty$ if $l_0 = 0$ so that normally incident waves would not be turned.

To determine $E(x)$, consider two parallel rays and let $A(x)$ be the distance between the two rays measured parallel to the y-axis. By the ray tube method

$$E(x)\frac{\partial\Omega}{\partial k}A(x) = \text{independent of } x$$

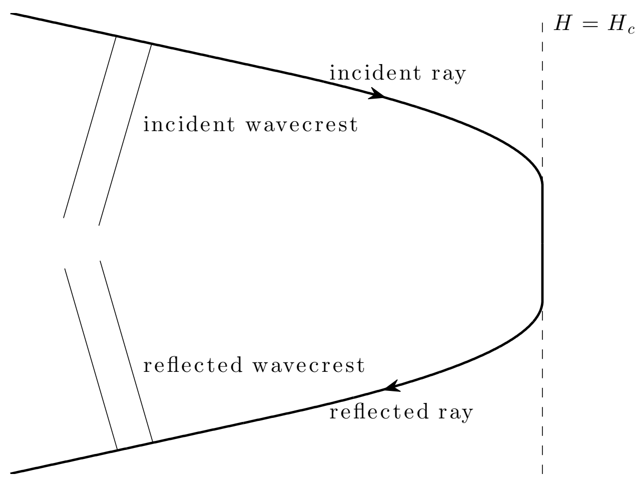
But on plane parallel contours, *i.e.,* $h(x)$, then the distance between two rays in the y direction must also independent of x, *i.e.,* $\mathrm{d}A/\mathrm{d}x = 0$. Thus

$$E(x)\frac{\partial\Omega}{\partial k}(x) = \text{const} \tag{5.46}$$

$$\Longrightarrow \qquad E(x) = E(0)\sqrt{\frac{h_0}{h(x)}\frac{\cos\theta_0}{\cos\theta}} \tag{5.47}$$

As waves propagate into shallower depth, $h \to 0$, then $E(x) \to (h_0/h)^{1/2} \cos\theta_0$ and energy increases with $h^{-1/2}$. However, for waves propagating into deeper water for increasing h, E becomes infinite at the caustic $h = h_c$ because $\cos\theta = 0$. The convergence of ray paths has squeezed a finite energy into an infinitesimal space.

Clearly the ray theory breaks down at the caustic, but it can be modified to make a useful prediction of the rest of the pattern. If no energy can pass $h = h_c$, then the energy flux past any x must vanish in the steady state. We therefore postulate a *reflected wave* with wavevector $(-k, l)$ whose rays also satisfy (5.33–5.38), but look like mirror images of the incident rays.



Now since $\mathbf{k}_{\text{incident}} = (k, l)$ while $\mathbf{k}_{\text{reflected}} = (-k, l)$, the resultant pattern is a *standing wave* in the x-direction and a *progressive wave* in the y-direction. The infinite E at the caustic remains an objectionable flaw of the ray theory, however. The next chapter re-considers this example using WKB theory and asymptotic matching. The results will show that the ray theory is essentially correct, and the breakdown at the caustic will be eliminated.

5.6 Homework

1. Show that for a slowly-varying medium, $\nabla \times \mathbf{k} = 0$.

2. For the image below qualititatively explain what the underlying depth contours must be and why the wave crests are bending as they do. Draw the ray lines. Where is wave energy more and less concentrated.

Chapter 6

A perfect fluid and acoustic waves

6.1 A perfect fluid (inviscid and compresible)

6.1.1 Equations

Note that here we neglect rotation. The conservation of mass and momentum equations for an inviscid and compressible (*i.e.,* perfect) fluid in a gravity field are.

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{u} = 0 \tag{6.1}$$

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p - \rho g\hat{\mathbf{k}}. \tag{6.2}$$

where velocity $\mathbf{u} \equiv (u, v, w)$ and $\hat{\mathbf{k}}$ is the unit upward vector. To close these two equations, we now need something that relates the pressure p and density ρ, such as an equation of state. This is relatively complex and requires a detailed examination of thermodynamics. Suffice to say that it is different for a gas and a liquid. In physical oceanography where things are assumed incompressible, an equation of state is often written as

$$\rho = \rho(p, T) \tag{6.3}$$

where T is the temperature of the fluid. Here are assuming that there is no salinity, which can be added to (6.3). This equation of state gives density ρ as a function of pressure p. We would like pressure as a function of density and other things. How to invert this relationship?

6.1.2 Basic Thermodynamics of a Perfect Fluid

In thermodynamics the change in internal energy E (units Joules) is written as

$$\mathrm{d}E = -p\,\mathrm{d}V + T\,\mathrm{d}\mathcal{S}$$

where $\mathcal{S}$ is the entropy. Rewritten per unit mass this equation becomes

$$\mathrm{d}e = -p\,\mathrm{d}\alpha + T\,\mathrm{d}s \tag{6.4}$$

where $\alpha = (1/\rho)$ and s is the specific entropy. Now assuming that the processes that we are examining are reversible. That is entropy does not increase. *Why is this reasonable based on the momentum equations?* For an reversible process, the specific entropy does not change $\mathrm{d}s = 0$, and so (6.4) can be re-written as

$$p = -\frac{\partial e(\rho, s)}{\partial \alpha} = \rho^2 \frac{\partial e(\rho, s)}{\partial \rho} = F(\rho, s) \tag{6.5}$$

where $p = F(\rho, s)$ is an inverted form of the equation of state. That is an inversion of (6.3). With entropy constant (isentropic process), we then have

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0 \tag{6.6}$$

and thus

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\partial F}{\partial \rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \frac{\partial F}{\partial s}\cancelto{0}{\frac{\mathrm{D}s}{\mathrm{D}t}} \tag{6.7}$$

which simplifies to an equation relating pressure evolution to density evolution,

$$\frac{\mathrm{D}p}{\mathrm{D}t} = c^2 \frac{\mathrm{D}\rho}{\mathrm{D}t} \tag{6.8}$$

where

$$c^2 \equiv \left.\frac{\partial p}{\partial \rho}\right|_s . \tag{6.9}$$

Note, c has units of $\mathrm{m\,s^{-1}}$, Now it is worth noting that we started with mass and momentum conservation equation (6.1,6.2) and now have now added a third constraint that is based on an internal energy conservation plus constant entropy (reversible or isentropic process) argument. The constant entropy was applied twice first to get (6.5) and second to get (6.8).

6.1.3 Simplest solution: the state of rest

It is always of interest to look at the simplest solution, which is a state of rest. This implies: $\partial/\partial t = 0$ and $\mathbf{u} = 0$. Thus,

$$0 = -\frac{\partial p}{\partial z} - \rho g \tag{6.10}$$

$$p = F(\rho, s) \tag{6.11}$$

Which is two equations for 3 unknowns (p, ρ, s). To technically get a unique solution one needs to specify an entropy profile. Such a solution may not be stable! However, this is esoteric compressible fluids and thermodynamics that is not very applicable to almost incompressible situations to the ocean and to the systems we study here.

6.2 Linearized dynamics of a perfect fluid

Now we will study linearized dynamics around the state of rest, where we assume there is a base (rest) and stable solution where $\mathbf{u}_0 = 0,\ p = p_0(z),\ \rho = \rho_0(z)$ and $s = s_0$ around which we will linearize.

Consider *small* perturbations around this rest state,

$$\mathbf{u} = 0 + \mathbf{u}' \tag{6.12}$$

$$p = p_0(z) + p' \tag{6.13}$$

$$\rho = \rho_0(z) + \rho', \tag{6.14}$$

where primed quantities are small. Note, below we are simply linearizing dimensional equations by throwing out terms quadratic in primed variables. Strictly speaking, one must linearize non-dimensional equations by perturbation expansion of a small parameter. Nevertheless, the linearized equations are

$$\frac{\partial}{\partial t}(\rho_o \mathbf{u}') = -\nabla p' - \rho' g \hat{\mathbf{k}} \tag{6.15}$$

$$\frac{\partial \rho'}{\partial t} = \nabla \cdot (\rho_0 \mathbf{u}') = 0 \tag{6.16}$$

$$\frac{\partial p'}{\partial t} + w' \frac{\mathrm{d}p_0}{\mathrm{d}z} = c^2 \left[\frac{\partial \rho'}{\partial t} + w' \frac{\mathrm{d}\rho_0}{\mathrm{d}z} \right] \tag{6.17}$$

Now using the fact that $\mathrm{d}p_0/\,\mathrm{d}z = -g\rho_0$ we can group the terms proportional to w' as

$$w' \left[c^2 \frac{\mathrm{d}\rho_0}{\mathrm{d}z} + g\rho_0 \right] = c^2 \frac{\rho_0}{g} w' \left[\frac{g}{\rho_0} \frac{\mathrm{d}\rho_0}{\mathrm{d}z} - \frac{g^2}{c^2} \right]$$

So now the linearized pressure equation becomes,

$$\frac{\partial p'}{\partial t} = c^2 \left[\frac{\partial \rho'}{\partial t} - \frac{\rho_0 w'}{g} N^2 \right], \tag{6.18}$$

$$N^2 \equiv \frac{-g}{\rho_0} \frac{\mathrm{d}\rho_0}{\mathrm{d}z} - \frac{g^2}{c^2} \tag{6.19}$$

where $c^2 = c^2(\rho_0, s_0)$ was defined in (6.9) above.

For those of you with previous Geophysical Fluid Dynamics experience, the quantity N^2 (6.19) may look familiar but also different that how it normally appears $N^2 = (-g/\rho_0)\frac{\mathrm{d}\rho_0}{\mathrm{d}z}$. We will explore in the next chapter. We will spend a fair amount of time with these linearized equations exploring both sound waves and internal gravity waves in the ocean but under different sets of assumptions.

6.3 Simplest waves: sound waves with negligible gravity

First, consider the case of no gravity $g = 0$. If gravity is negligible, then it follows from (6.19) that stratification $N^2 = 0$ is a reasonable assumption. Then p_0 is constant from (6.10) The linearized equations then become

$$\frac{\partial}{\partial t}(\rho_0 \mathbf{u}') = -\nabla p' \tag{6.20}$$

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 \mathbf{u}') = 0 \tag{6.21}$$

$$\frac{\partial p'}{\partial t} = c^2 \frac{\partial \rho'}{\partial t} \tag{6.22}$$

Substituting (6.22) into (6.21), leads to

$$\frac{\partial p'}{\partial t} + c^2 \nabla \cdot (\rho_0 \mathbf{u}') = 0 \tag{6.23}$$

and the taking a time derivative of (6.23) and subsituting (6.20) yields

$$\frac{\partial^2 p'}{\partial t^2} = c^2 \nabla^2 p' \tag{6.24}$$

This is again the 2nd-order wave equation or more commonly the *wave equation* we examined in Chapter 1.

Now suppose c^2 is a constant, we already know from Chapter 1 what the solution is and how to solve it. Again for review, consider one dimension:

$$p'_{tt} = c^2 p'_{xx} \iff \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) p' = 0$$
$$\implies p' = f(x + ct) + g(x - ct).$$

In three dimensions this means (by rotational symmetry) that the solution looks like

$$p' = f(lx + my + nz + ct) + g(lx + my + nz - ct).$$

where $l^2 + m^2 + n^2 = 1$.

6.3.1 Plane waves

Motivated by our knowledge of Fourier decomposition, lets now plug in a plane wave solution

$$p' = \hat{p} e^{\mathrm{i}(kx + ly + mz - \omega t)}$$

into (6.24) and with substitution implies,

$$\omega^2 = c^2(k^2 + l^2 + m^2) \qquad \text{(dispersion relation)}$$
$$\omega > 0 \qquad \text{wave propagates in the direction of } \mathbf{k}$$
$$\omega < 0 \qquad \text{wave propagates opposite the direction of } \mathbf{k}$$

where k, l, and m are the wavenumbers in the direction of the vector wavenumber $\mathbf{k}$. The phase speed c of these acoustic waves is given by

$$c^2 = \frac{\omega^2}{|\mathbf{k}|^2} \equiv \frac{\omega^2}{(k^2 + l^2 + m^2)}$$

So it is not a coincidence that we used the symbol c for $\partial p/\partial\rho|_s$ earlier (6.9).

Now plugging in the plane wave solution for each of the mass, momentum and pressure evolution equations we get,

$$\begin{aligned} \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}' &= 0 & & & -\mathrm{i}\omega\hat{\rho} + \mathrm{i}\rho_0 \mathbf{k}\cdot\hat{\mathbf{u}} &= 0 \\ \rho_0 \frac{\partial \mathbf{u}'}{\partial t} &= -\nabla p' & \Longrightarrow & & -\mathrm{i}\rho_0\omega\hat{\mathbf{u}} &= -\mathrm{i}\mathbf{k}\hat{p} \\ \frac{\partial p'}{\partial t} &= c^2 \frac{\partial \rho'}{\partial t} & & & -\mathrm{i}\omega\hat{p} &= -\mathrm{i}\omega c^2 \hat{\rho} \end{aligned}$$

which reduces to

$$\hat{\mathbf{u}} = \frac{\mathbf{k}}{\rho_0 \omega}\hat{p} \tag{6.25}$$

$$\hat{\rho} = \frac{1}{c^2}\hat{p} \tag{6.26}$$

$$\tag{6.27}$$

This means that

1. Similar to surface gravity wave, once we have a solution for pressure p' and we know $\mathbf{k}$, we know velocity $\mathbf{u}'$ and density ρ' fluctuations.

2. Velocity $\mathbf{u}'$ is in the direction of $\mathbf{k}$ and so is perpendicular to wave crests.

3. All variables are in phase. This is in contrast with surface gravity waves where η and w were $\pi/4$ out of phase.

6.3.2 Neglecting gravity justified?

Is gravity really negligible? By examining the sizes of the neglected terms in (6.18) above, we can conclude : yes, provided that the vertical wavenumber m satisfies that

$$m \gg \frac{g}{c^2}, \quad \text{and} \quad m \gg \frac{\tilde{N}^2}{g}, \tag{6.28}$$

where $\tilde{N}^2 = -(g/\rho_0)\mathrm{d}\rho_0/\mathrm{d}z$, or equivalently that

$$\text{wavelength} \ll \frac{c^2}{g} \quad , \quad \frac{g}{\tilde{N}^2}$$

It can be shown that these two inequalities are the general condition that sound waves and internal gravity waves decouple. (see Lighthill) and this will come up again. In the problem set, we will explore this inequality further.

6.3.3 Plane waves: Reflection off of a solid boundary

Consider a plane wave in two dimensions incident upon a solid boundary at $z = 0$ with angle θ_I. This means we have an incident wave of the form

$$p'_I = \hat{p}_I e^{\mathrm{i}(kx+mz-\omega t)} \tag{6.29}$$

where $k/m = \tan\theta_I$. The boundary condition at the solid boundary is $\mathbf{u}\cdot\mathbf{n} = 0$ or in this case simply $w = 0$. From the momentum equation, if $w = 0$ at $z = 0$, then $\partial p/\partial z = 0$ at $z = 0$. To satisfy this boundary condition, we must add a reflected wave of the form

$$p'_R = \hat{p}_R e^{\mathrm{i}(kx-mz-\omega t)} \tag{6.30}$$

as at $z = 0$,

$$(p'_R + p'_I)_z = m(\hat{p}_R - \hat{p}_I)e^{\mathrm{i}(kx-\omega t)} = 0,$$

thus, $\hat{p}_R = \hat{p}_I$ and the solution is

$$p' = p'_I + p'_R = 2\hat{p}e^{\mathrm{i}(kx-\omega t)}\cos(mz) \tag{6.31}$$

Satisfying the boundary condition. As the reflected wave has the same $|\mathbf{k}|$ but opposite signed vertical wavenumber m, we see that $\theta_\mathrm{R} = -\theta_\mathrm{I}$. This property applies to reflection off of any solid surface and is called *specular* reflection.

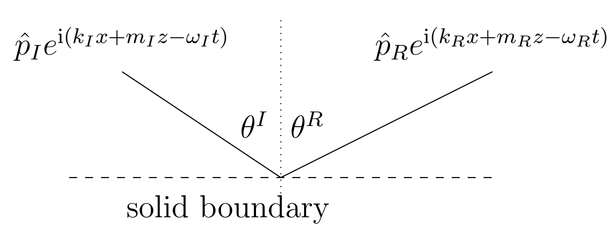


Figure 6.1: Reflection of a sound wave on a solid boundary.

6.4 Problem Set

1. Consider the assumption/simplification regarding acoustic waves being reversible, which implies constant entropy $\mathrm{D}s/\mathrm{D}t = 0$. What approximation was applied to arrive at the perfect fluid momentum equation that is consistent with the constant entropy assumption?

2. Consider a plane sound wave in the atmosphere is incident upon a resting flat ocean surface at $z = 0$ (Figure. 6.2. The angle of incidence is θ_I. Compute the direction and pressure amplitudes of the reflected and transmitted waves. Treat the two fluids as having constant densities of ρ_a and ρ_w for air and water and and sound speeds c_a and c_w. Neglect gravity. HINT: pressure and normal velocity must be continous at $z = 0$.

3. Using these results, calculate the air to water transmitted pressure and angle for an incident acoustic wave of $\theta = 10°$. Use $\rho_\mathrm{a} = 1$ kgm^{-3}, $\rho_\mathrm{w} = 10^3$ kgm^{-3}, $c_\mathrm{a} = 350$ m s^{-1}, and $c_\mathrm{w} = 1500$ m s^{-1}.

4. Is there a critical angle where air-to-water transmission stops and is reflected?

5. Repeat question 3 from the ocean to the atmosphere. What conclusions can you draw for sound transmission between the ocean and atmosphere?

6. What frequency range to marine mammals typically use?

7. Show that the effect of gravity in sound waves can be neglected if the inequalities (6.28) are true using the plane wave solution in (6.18).

8. For a typical marine mammal frequency range, is the neglect of gravity valid? Use

$$c = 1.5\,\mathrm{km/sec} \qquad\qquad N^2 = (3 \times 10^{-3}\mathrm{sec}^{-1})^2$$

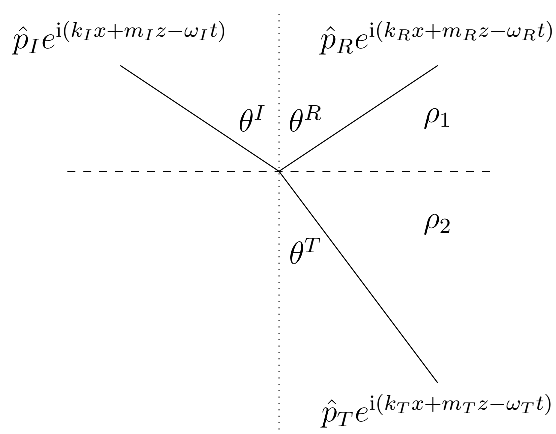


Figure 6.2: Sound wave relection and transmission at a air/water boundary.

Chapter 7

Energy conservation in perfect fluid and acoustic waves

7.1 Energy conservation in a perfect fluid

As with surface gravity waves, we now seek an energy conservation equation for both the "perfect" fluid and for linearized sound waves. First we consider a perfect fluid. Let us now consider a fluid enclosed by a boundary. What is the expected energy per unit volume? schematically energy is partitioned into three components,

$$\begin{array}{ccccc} \text{kinetic} & + & \text{gravitational potential} & + & \text{internal} \\ \rho\frac{1}{2}\mathbf{u}\cdot\mathbf{u} & + & \rho g z & + & \rho e(\rho, s). \end{array}$$

The internal energy comes from both the compressibility of the fluid and the heat content. Recall from thermodynamics and the previous lecture that the internal energy of a fluid has $\mathrm{d}E = -p\,\mathrm{d}V + T\,\mathrm{d}S$ and per unit mass (with units of $\mathrm{J\,kg^{-1}}$) is $\mathrm{d}e = p\,\mathrm{d}\alpha + T\,\mathrm{d}\eta$ where $\alpha = \rho^{-1}$. Because e is per unit mass, we need to multiply by ρ so that all terms have units $\mathrm{J\,m^{-3}}$.

Now integrating over a fluid volume we get the total energy E_{TOT} (units J),

$$E_{\mathrm{TOT}} = \iiint \rho\left[\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + gz + e(\rho, s)\right]\mathrm{d}V$$

and taking a time-derivative to get an energy conservation equation

$$\frac{\mathrm{d}E_{\mathrm{TOT}}}{\mathrm{d}t} = \iiint \rho\frac{D}{Dt}\left[\qquad\qquad\qquad\qquad\right]\mathrm{d}V = \,? \tag{7.1}$$

To evaluate this we begin by forming a kinetic energy term from the momentum equation

(6.2)

$$\rho\frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) = -\mathbf{u}\cdot\nabla p - \rho g w$$
$$= -\nabla\cdot(\mathbf{u}p) + p(\nabla\cdot\mathbf{u}) - \rho g w$$
$$= -\nabla\cdot(\mathbf{u}p) - \frac{p}{\rho}\frac{D\rho}{Dt} - \rho g w$$

Here, we used the perfect fluid conservation of mass equation (6.1),

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{u} = 0$$

to transform $p(\nabla\cdot\mathbf{u}) = -(p/\rho)D\rho/Dt$. Now recall from the equation of state for pressure (6.5) - assuming constant entropy - that

$$p = \rho^2\frac{\partial e(\rho,s)}{\partial\rho}, \tag{7.2}$$

and so we can re-write the middle term as

$$= -\nabla\cdot(\mathbf{u}p) - \rho\frac{\partial e}{\partial\rho}(\rho,s)\frac{D\rho}{Dt} - \rho g w$$
$$= -\nabla\cdot(\mathbf{u}p) - \rho\frac{\mathrm{D}e(\rho,s)}{\mathrm{D}t} - \rho g w$$

where the last step is because

$$\frac{\partial e}{\partial\rho}(\rho,s)\frac{D\rho}{Dt} = \frac{\mathrm{D}e(\rho,s)}{\mathrm{D}t}.$$

The potential energy term $-\rho g w$ can also be rewritten as $-\rho g Dz/Dt$ (Why?). Thus, we can now write the kinetic energy evolution equation as

$$\rho\left(\frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \frac{\mathrm{D}(gz)}{\mathrm{D}t} + \frac{\mathrm{D}e(\rho,s)}{\mathrm{D}t} +\right) = -\mathbf{u}\cdot\nabla p. \tag{7.3}$$

or

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + gz + e(\rho,s)\right) = -\mathbf{u}\cdot\nabla p. \tag{7.4}$$

This (7.4) means that the material derivative of the specific (per unit volume) total energy ($\mathrm{J\,m^{-3}}$) changes from the local work ($-\mathbf{u}\cdot\nabla p$) done upon it. By volume integrating (7.4), we can now rewrite the total energy evolution on the volume as

$$\frac{\mathrm{d}E_{\mathrm{TOT}}}{\mathrm{d}t} = -\iiint \nabla\cdot(\mathbf{u}p)\,\mathrm{d}V = \iint p(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}A \tag{7.5}$$

such that the total energy of the perfect fluid is changed by the work done on the fluid at the boundaries. If no work is done on the fluid, energy can still move from kinetic to potential to internal and back reversibly because in this system we have assumed constant entropy. *Question: For a perfect fluid enclosed in a solid boundary, what is the time evolution of E_{TOT}?*

7.2 Energy conservation in full linearized equations without gravity and constant background density

Now let us examine the linearized equation with no gravity ($g = 0$) and constant background density $\mathrm{d}\rho_0/\mathrm{d}z = 0$ (or $N^2 = 0$). This means there is no potential energy.

$$\begin{aligned} \rho_0 \frac{\partial \mathbf{u}'}{\partial t} &= -\nabla p' && \text{momentum} \\ \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}' &= 0 && \text{continuity} \\ \frac{\partial p'}{\partial t} &= c^2 \frac{\partial \rho'}{\partial t} \end{aligned}$$

Forming an energy equation by multiplying linearized momentum by $\mathbf{u}'$ gives

$$\begin{aligned} \rho_0 \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}' \cdot \mathbf{u}'\right) &= -\mathbf{u}' \cdot \nabla p' \\ &= -\nabla \cdot (\mathbf{u}'p') + p'(\nabla \cdot \mathbf{u}') \\ &= -\nabla \cdot (\mathbf{u}'p') + p'\left[-\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t}\right] \\ &= -\nabla \cdot (\mathbf{u}'p') - \frac{p'}{\rho_0}\left[\frac{1}{c^2}\frac{\partial p'}{\partial t}\right] \\ &= -\nabla \cdot (\mathbf{u}'p') - \frac{\partial}{\partial t}\left[\frac{1}{2\rho_0 c^2} p'^2\right] \end{aligned}$$

Thus the specific energy conservation equation becomes,

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho_0 \mathbf{u} \cdot \mathbf{u} + \frac{1}{2\rho_0 c^2} p'^2\right] = -\nabla \cdot (\mathbf{u}'p')$$

which is similar to the full energy conservation equation derived in the previous section.

Specifically, we can align these terms by showing that the form of the energy conserved by the linearized equations "makes sense". Obviously, for the kinetic energy, $\frac{1}{2}\rho_0 \mathbf{u}' \cdot \mathbf{u}'$ is a logical approximation to $\frac{1}{2}\rho \mathbf{u} \cdot \mathbf{u}$. For the internal energy term, it appears the linearized pressure squared term,

$$\frac{1}{2\rho_0 c^2}(p')^2$$

is an approximation to internal energy $\rho e(\rho, s)$.

We can explain this as follows. The change in the internal energy per unit mass caused by the fluctuating pressure p' is

$$\mathrm{d}e = -p'\,\mathrm{d}\alpha$$

Now recall $\alpha = (\rho_0 + \rho')^{-1}$ where ρ_0 is a constant and all density fluctuations (and thus α fluctuations) are in ρ'. Then

$$\begin{aligned} \mathrm{d}\alpha &= -\,(\rho_o + \rho')^{-2}\,\mathrm{d}\rho' \\ &= -\,\rho_o^{-2}\,\mathrm{d}\rho' \end{aligned}$$

Thus, using the fact that $\mathrm{d}p' = c^2\,\mathrm{d}\rho'$, we get

$$\begin{aligned} \mathrm{d}e = -p'\,\mathrm{d}\alpha &= \frac{p'}{\rho_0^2}\,\mathrm{d}\rho' = \frac{p'}{\rho_0^2 c^2}\,\mathrm{d}p' \\ &= \mathrm{d}\left(\frac{1}{2\rho_0^2 c^2}(p')^2\right), \end{aligned}$$

in agreement with the previous section. This reasoning not generally applicable, but it works for these simplified conditions: reversible, $g = 0$, and $N^2 = 0$.

7.3 Energy and energy flux in sound waves

We have already considered plane wave solutions of the linearized equations (see Chapter 6.3.1),

$$\mathbf{u}' = \mathrm{Re}(\hat{\mathbf{u}}e^{\mathrm{i}\theta}) \text{ and } p' = \mathrm{Re}(\hat{p}e^{\mathrm{i}\theta}) \quad , \quad \theta \equiv \mathbf{k}\cdot\mathbf{x} - \omega t$$

with

$$\hat{\mathbf{u}} = \frac{\mathbf{k}}{\rho_0\omega}\hat{p}$$

where $\hat{p}$ and $\hat{\mathbf{u}}$ are real numbers.

As shown above, these solutions of the linearized equations obey the linearized energy conservation law

$$\frac{\partial}{\partial t}\mathcal{E} = -\nabla\cdot\mathbf{F}$$

where $\mathcal{E}$ is the specific energy and $\mathbf{F}$ is the (sound) wave energy flux,

$$\mathcal{E} = \frac{1}{2}\rho_0\mathbf{u}'\cdot\mathbf{u}' + \frac{1}{\rho_0 c^2}\frac{(p')^2}{2} \tag{7.6}$$

$$\mathbf{F} = \mathbf{u}'p' \tag{7.7}$$

Let $\langle\rangle$ denote the average over a wavelength or period and averaging over a wave period we find that

$$\langle\mathcal{E}\rangle = \frac{1}{2}\frac{1}{\rho_0 c^2}\hat{p}^2. \tag{7.8}$$

The average acoustic energy flux is

$$\langle \mathbf{F} \rangle = \langle \mathbf{u}' p' \rangle = \left\langle \left(\frac{\mathbf{k}}{\rho_0 \omega} \hat{p} \cos\theta \right) (\hat{p} \cos\theta) \right\rangle = \frac{\mathbf{k}}{\rho_0 \omega} \frac{1}{2} \hat{p}^2 \tag{7.9}$$

Thus $\langle \mathbf{F} \rangle$ points in the direction of wave propagation. *Question: Can one write the average energy flux* $\langle \mathbf{F} \rangle$ *as a product of a velocity times* $\langle \mathcal{E} \rangle$*? What would that velocity be?*

7.4 Problem Set

1. Using the definition of the material derivative D/Dt rationalize how w can be written as Dz/Dt in

2. For a perfect fluid enclosed in a solid, non-moving, and adiabatic boundary, if at $t = 0$ $E_{\text{TOT}} = E_0$, what is the time-evolution evolution of E_{TOT}?

3. In the average acoustic energy density equation (7.8), what is the ratio of kinetic to internal energy? First, apply intuitive reasoning and then demonstrate the answer from time-averaging (7.6) with the plane wave solution.

4. Using the acoustic energy flux (7.9), rewrite it so that $\langle \mathbf{F} \rangle = \text{velocity} \times \langle \mathcal{E} \rangle$. What is the corresponding velocity? How does it relate to other wave energy equation systems you have studied?

Chapter 8

The Boussinesq Approximation

8.1 Physical interpretation of the Väisälä frequency N

In the previous lecture on acoustic waves, we derived a quantity that was called N^2 and was given by (6.19),

$$N^2 \equiv \frac{-g}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z} - \frac{g^2}{c^2}.$$

The quantity N is called the Väisälä frequency. It is also often called the *buoyancy* frequency. It plays an important role in much of waves and turbulence in stratified fluids. Here, its meaning will be examined and then we will look at approximations to it.

But first we re-derive the Väisälä (or buoyancy) frequency in another context. A particle displaced a vertical distance δz from its rest state at z_0 of p_0 and ρ_0 experiences a density change of

$$\delta\rho_{\mathrm{part}} = \frac{1}{c^2}\delta p_{\mathrm{part}} = -\frac{\rho_0 g}{c^2}\delta z \tag{8.1}$$

provided that the pressure of the water particle remains the same as the surrounding fluid. The displaced particle experiences a buoyancy force per unit volume of

$$\rho_{\mathrm{part}}\frac{\mathrm{d}w_{\mathrm{part}}}{\mathrm{d}t} - (\rho_{\mathrm{part}} - \rho)g = 0$$

where $\rho_{\mathrm{part}} = (\rho_0 - \rho_0 g/c^2)\delta z$ from (8.1). For small displacements, the density of the surrounding fluid

$$\rho = \rho_0 + \frac{\mathrm{d}\rho_0}{\mathrm{d}z}\delta z, \tag{8.2}$$

as our base solution had $\rho_0 = \rho_0(z)$. Thus, with $w_{\mathrm{part}} = d/dt(\delta z)$ a harmonic oscillator equation can be written for δz,

$$\begin{aligned}\rho_{\text{part}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta z &= -(\rho_{\text{part}}-\rho)g\\ &\approx \left[\left(\frac{\rho_0(z)g}{c^2}\delta z\right)-\left(\frac{\mathrm{d}\rho_0}{\mathrm{d}z}\delta z\right)\right]g\\ &= \left[\frac{g^2\rho_0(z)}{c^2}-g\frac{\mathrm{d}\rho_0(z)}{\mathrm{d}z}\right]\delta z\end{aligned}$$

Thus this harmonic oscillator equation can be written as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta z = -N^2\delta z \tag{8.3}$$

where the Väisälä frequency squared is defined as earlier (6.19)

$$N^2 = \left[-\frac{g}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}-\frac{g^2}{c^2}\right]$$

The solution to (8.3) is $e^{\pm iNt}$. If $N^2 < 0$, then the fluid is unstable as the particle displaces exponentially. If $N^2 > 0$, the displaced particle tends to return to its initial location, and oscillates about it at frequency N. The significance of $N^2 < 0$ could also be anticipated from the fact that $N^2 < 0$ makes the linearized energy a non-definite quadratic form.

In section 7.2, we examined energy conservation for the linearized terms without gravity. Here, this can now be revisited. Now, revert to the notation $w' = \frac{\mathrm{d}}{\mathrm{d}t}\delta z$ and such that equation (8.3) is rewritten to

$$\frac{\mathrm{d}w'}{\mathrm{d}t} = -N^2\delta z$$

which implies that by multiplying by w'

$$w'\frac{\mathrm{d}w'}{\mathrm{d}t} = -N^2\delta z'\frac{\mathrm{d}\delta z}{\mathrm{d}t} \tag{8.4}$$

thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}(w')^2+\frac{N^2}{2}(\delta z)^2\right) = 0 \tag{8.5}$$

The second term is now the linearized potential energy to go with the linearized kinetic and internal energy derived in section 7.2.

8.2 Separation of Density: Boussinesq Approximation

The density of seawater is never far from $1\,\mathrm{gm/cm^3}$ or 10^3 kg m^{-3}. This observation is the basis of the Boussinesq approximation which will simplify the equations of the perfect fluid, making the resulting fluid incompressible, and filter out sound waves.

Consider the at rest solution and write the fluid density as,

$$\rho(x,y,z,t) = \underbrace{\rho_{00} + \rho^*(z)}_{\rho_0(z)} + \rho'(x,y,z,t)$$

and

$$p(x,y,z,t) = p_0(z) + p'(x,y,z,t)$$

As before, the vertical momentum equation reduces to the hydrostatic balance

$$\frac{\mathrm{d}p_0(z)}{\mathrm{d}z} = -\rho_0(z)g.$$

In the Boussinesq approximation, one assumes that

$$|\rho^*(z)| \quad , \quad |\rho'| \ll \rho_{00}$$

Typical sizes are

$$\rho_{00} \sim 1030 \qquad \rho^* \sim 20 \qquad \rho' \sim 3 \text{ kg m}^{-3}$$

Note ρ^* chiefly represents density variation caused by pressure changes

$$|\rho^*| \sim \frac{\Delta p}{c^2} = \frac{\rho_{00} g d}{c^2}$$

where d is a scale depth. Thus, $\quad |\rho^*| \ll \rho_{00} \quad$ if $\boxed{d \ll \frac{c^2}{g} \equiv \text{scale depth}}$. Note that ρ^* could also incorporate the average temperature and salinity changes in the vertical.

Note that as defined here, $\rho' = \rho'(x,y,z,t)$ can have horizontal, vertical, and temporal fluctuations that are linked to temperature or salinity variations. Thus

$$\begin{aligned}
|\rho'| &\sim \text{(observed range of density variation)} - \text{(the part accounted for by pressure)} \\
&\sim \left(-\frac{\partial \rho_0}{\partial z} d\right) - \left(\left.\frac{\partial \rho}{\partial p}\right|_s \frac{\partial p}{\partial z} d\right) \\
&\sim \left(-\frac{\partial \rho_0}{\partial z} + \frac{1}{c^2}(-\rho_{00} g)\right) d \\
&\sim \frac{\rho_{00}}{g}\left[\frac{-g}{\rho_{00}}\frac{\partial \rho_0}{\partial z} - \frac{g^2}{c^2}\right] d = \frac{\rho_{00}}{g} N^2 d
\end{aligned}$$

Thus $\quad |\rho'| \ll \rho_{00} \quad$ if $\quad \boxed{d \ll \frac{g}{N^2}}$

8.3 Application to the Perfect Fluid Equations

Exact momentum

$$\rho_{00}\left[1+\frac{\rho^*}{\rho_{00}}+\frac{\rho'}{\rho_{00}}\right]\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p' - \rho' g\hat{\mathbf{k}}$$

becomes

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla\left(\frac{p'}{\rho_{00}}\right) - \frac{\rho' g}{\rho_{00}}\hat{\mathbf{k}} \tag{8.6}$$

Exact continuity is

$$\frac{\partial \rho'}{\partial t} + \mathbf{u}\cdot\nabla(\rho^* + \rho') + (\rho_{00} + \rho^* + \rho')\nabla\cdot\mathbf{u} = 0$$

and after dividing by ρ_{00} one gets

$$\frac{1}{\rho_{00}}\frac{\partial \rho'}{\partial t} + \mathbf{u}\cdot\nabla\left(\frac{\rho^*+\rho'}{\rho_{00}}\right) + \left(1+\frac{\rho^*+\rho'}{\rho_{00}}\right)\nabla\cdot\mathbf{u} = 0$$

which reduces to the continuity equation for an incompressible fluid

$$\nabla\cdot\mathbf{u} = 0 \tag{8.7}$$

The exact thermodynamic equation

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{1}{c^2}\frac{\mathrm{D}p}{\mathrm{D}t} \tag{8.8}$$

becomes

$$\frac{\mathrm{D}\rho'}{\mathrm{D}t} - \frac{w\rho_{00}}{g}N^2 = \frac{1}{c^2}\frac{\mathrm{D}p'}{\mathrm{D}t}. \tag{8.9}$$

In the type of motion we consider next, the pressure gradient force is of the same size as the buoyancy force. Thus $p' \approx \rho' g d$, where d is the vertical displacement. Thus,

$$\frac{\mathrm{D}\rho'}{\mathrm{D}t} \gg \frac{1}{c^2}\frac{\mathrm{D}p'}{\mathrm{D}t}$$

if $d \ll c^2/g$, a constraint we have seen before and is also almost completely valid in the ocean. Then the thermodynamics equation becomes something else: essentially a perturbation density conservation equation,

$$\frac{\mathrm{D}\rho'}{\mathrm{D}t} - \frac{w\rho_{00}}{g}N^2 = 0 \tag{8.10}$$

This is analogous to a tracer conservation equation.

8.4 Almost Primitive Equations

The 3 equations that result from the Boussinesq approximation are given here again as a group

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho_{00}}\nabla p' - \frac{\rho' g}{\rho_{00}}\hat{\mathbf{k}} \tag{8.11}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{8.12}$$

$$\frac{\mathrm{D}\rho'}{\mathrm{D}t} - \frac{w\rho_{00}}{g}N^2 = 0 \tag{8.13}$$

where now the buoyancy frequency for an incompressible Boussinesq fluid is now written as

$$N^2(z) = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_0(z)}{\mathrm{d}z}. \tag{8.14}$$

which is the *typical* form of N^2 that we are familiar with - in contrast with (6.19), and is due to the application of the Boussinesq approximation.

Note that these equations are *not* hydrostatic. But they are incompressible $\nabla \cdot \mathbf{u} = 0$. These equations (8.11) *with* the hydrostatic assumption are often called the *primitive equations. QUESTION: What is the hydrostatic assumption and how would (8.11) be rewritten?*

These equation can be written to a even simpler form if we define buoyancy b as

$$b \equiv -\frac{\rho' g}{\rho_{00}} \tag{8.15}$$

and define a density-normalized pressure as $\phi = p/\rho_{00}$ then,

$$\boxed{\begin{aligned} \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} &= -\nabla\phi + b\hat{\mathbf{k}} \\ \nabla \cdot \mathbf{u} &= 0 \\ \frac{\mathrm{D}b}{\mathrm{D}t} + wN^2 &= 0 \end{aligned}} \tag{8.16}$$

Note, we are neglecting rotation here and the term $f\hat{\mathbf{k}}\times\mathbf{u}$ (where f is the Coriolis parameter) can be added to LHS of the momentum equation. We will do that later. Note that previously the mass and internal energy equations were linked in that both involved $\mathrm{D}\rho/\mathrm{D}t$. Now with the Boussinesq approximation, density ρ or buoyancy $b = -g\rho'/\rho_{00}$ are tracers that are advected around the ocean. Their dynamical influence comes in only in the gravity term of the momentum equations. In this sense density or buoyancy are *active* tracers. These equations (8.16) give rise to a range of solutions including internal gravity waves which will be discussed in the next chapter.

8.4.1 Equation of State: density and temperature

Furthermore, on ocean applications with salinity and temperature, we now see a more traditional form of the equation of state where density variations ρ' are due to temperature T' and salinity variations. Here we will consider a fluid where density depends only on temperature,

$$\rho = \rho(T). \tag{8.17}$$

As it is only the small density fluctuations ρ' that are dynamically significant, we only need to relate ρ' to T'.

$$\rho' = \rho - \rho_0 = \frac{\mathrm{d}\rho_0(T_0)}{\mathrm{d}T} T' \tag{8.18}$$

$$\rho' = -\alpha T' \tag{8.19}$$

$$\alpha = -\frac{\mathrm{d}\rho_0(T_0)}{\mathrm{d}T} \tag{8.20}$$

where the compressibility effects of pressure on density are ignored *Why?*

Note, with this equation of state, we can see that buoyancy b is a bit like temperature T. *Can you rewrite the equations (8.16) in the form using T' instead?*

8.5 Acoustic and Internal Gravity Wave Decomposition

For acoustic waves in the ocean we neglected the effect of gravity. With the Boussinesq approximation, we end up with a compressible fluid where density variations only influence momentum through the gravity term. The general linearized perfect fluid equations are:

$$\frac{\partial \mathbf{u}'}{\partial t} = -\frac{1}{\rho_{00}}\nabla p' - \frac{\rho' g}{\rho_{00}}\hat{\mathbf{k}}$$

$$\frac{\partial \rho'}{\partial t} + \rho_{00}\nabla \cdot (\mathbf{u}') = 0$$

$$\frac{\partial \rho'}{\partial t} - \frac{w\rho_{00}N^2}{g} = \frac{1}{c^2}\frac{\partial p'}{\partial t}$$

Terms in blue appear only in the equations for internal gravity waves, and terms in red appear only in the equations for sound waves. The terms in black are in both sets of equations.

Each of the two subsets of equations are valid for the type of motion to which they apply, provided that

$$d \ll \frac{c^2}{g}, \quad \text{and} \quad d \ll \frac{g}{N^2}.$$

This is confirmed by Lighthill (chap 3) using the straightforward (but very algebraic) approach of linearizing the full equations and showing that the sound and gravity waves are decoupled if these inequalities apply.

8.6 The Boussinesq approximation: Take 2

The formalism of the Boussinesq approximation can seem a bit bewildering. Why is this being done? What is it good for? Without the benefit of knowing everything that this enables in terms of Geophysical Fluid Dynamics, it is hard to get the point. Here, a take 2 explanation is given.

Recall the mass, momentum, and pressure equations of a perfect fluid (6.1, 6.2, 6.8) are

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{u} = 0$$
$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p - \rho g\hat{\mathbf{k}}$$
$$\frac{\mathrm{D}p}{\mathrm{D}t} = c^2\frac{\mathrm{D}\rho}{\mathrm{D}t}.$$

These 3 equations are nice because it is 3 equations with 3 unknowns. Although not a mathematical proof, it is satisfying that one can utilize these equations for useful means. However, these are complicated nonlinear equations even if c^2 is a constant.

For sound waves, we assumed that gravity was negligible ($g = 0$) and c^2 was constant, and then linearized about a state of rest, which threw out all other nonlinearity yielding a nice set of equations (6.20–6.22),

$$\frac{\partial}{\partial t}(\rho_0\mathbf{u}') = -\nabla p'$$
$$\frac{\partial\rho'}{\partial t} + \nabla(\rho_0\mathbf{u}') = 0$$
$$\frac{\partial p'}{\partial t} = c^2\frac{\partial\rho'}{\partial t}$$

These equations could be relatively straightforwardly solved.

Now for other GFD type motions, one could simply assert that the ocean (or atmosphere or Saturn's atmosphere or a star's plasma) is incompressible, *i.e.*, $\nabla\cdot\mathbf{u} = 0$. This is a gigantic simplification and often is simply asserted. It is often practically true but remember the ocean (and atmosphere) has increased near-bed density in the abyss due to compression effects.

With $\nabla\cdot\mathbf{u} = 0$, this now becomes the statement of mass conservation. We are now left with momentum conservation

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p - \rho g\hat{\mathbf{k}} \tag{8.21}$$

which is still pretty damn nonlinear - in some places cubic, that is $\rho u u_x$. This is very difficult to deal with both analytically and numerically. And it can obscure the physics. It could be re-written as

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\rho^{-1}\nabla p - g\hat{\mathbf{k}} \tag{8.22}$$

which is the same thing.

Aside: Now, consider for a second uniform density $\rho = \rho_0$. Is it weird that there are two equations for two unknowns ($\mathbf{u}$ and p) but only an evolution equation for $\mathbf{u}$? How is one supposed to solve for p? Hmmmm.

Now back to (8.21), we want to get rid of some nonlinearity, particularly off of the density. If we say, gosh I believe that for many GFD flows of interest, density effects will only come in with gravity and not in the momentum equation. The fluctuations of density are small. So $\rho \mathrm{D}\mathbf{u}/\mathrm{D}t \approx \rho_0 \mathrm{D}\mathbf{u}/\mathrm{D}t$ and then we divide through by ρ_0 to get

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\rho_0^{-1}\nabla p - \frac{\rho g}{\rho_0}\hat{\mathbf{k}}$$

Now variations in density only affect the gravity terms and density is linear! This is much much easier to deal and is in practice used in all of non-acoustic oceanic GFD.

8.7 Problem Set

1. Why is it ok to ignore the effects of varying pressure on density in (8.17) and (8.18)?

2. Assuming that density is a linear function of temperature and that, like density, temperature $T = \bar{T}(z) + T'(x, y, z, t)$, rewrite the equations (8.16) using T' and $\bar{T}$ instead of b and N^2 as the density-related variable.

3. For an ocean where density only depends on temperature, imagine you did a CTD cast and get a $T(z)$ profile. How would you estimate $N^2(z)$?

4. If the ocean density depends on salinity AND temperature what else would you have to do or *add* to these equations? Assume salinity $S = \bar{S}(z) + S'(x, y, z, t)$. Recall that in the ocean temperature monotonically decreases with depth but salinity doesn't always.

5. What is the hydrostatic assumption and how would the momentum equation in (8.11) be rewritten under the hydrostatic assumption. Hint, you have to write the horizontal and vertical equations momentum equations separately.

Note: it can be fun to play with an equation of state and helpful for question #1. A modern MATLAB toolbox for this at http://www.teos-10.org/software.htm

Chapter 9

Linear Internal Waves

Here we now derive linear internal waves in a Boussinesq fluid with a steady (not time varying) stratification represented by bouyancy frequency $N^2(z) = -(g/\rho_0)\mathrm{d}\rho/\mathrm{d}z$ and no rotation $f = 0$.

9.1 Linearized Equations

Starting with (8.16), the linearized Boussinesq equations are:

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla\phi + b\hat{\mathbf{k}} \tag{9.1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{9.2}$$

$$\frac{\partial b}{\partial t} + wN^2 = 0 \tag{9.3}$$

where recall the stratification is represented by the buoyancy frequency squared (6.19)

$$N^2(z) = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_0(z)}{\mathrm{d}z}.$$

and $b = -g\rho'/\rho_0$. Recall also that we can think of buoyancy b as a kind of "temperature".

9.2 Linear Vertical Velocity Equation

To get to simple internal waves, we derive an equation for the vertical velocity from the linearized equations by looking for oscillating solutions, *i.e.*, proportional to $e^{-\mathrm{i}\omega t}$, for steady but potentially vertically-varying $N^2(z)$. Let us first assume that all variables are proportional to $e^{-\mathrm{i}\omega t}$, *i.e.*, $u(x,y,z,t) = \hat{u}(x,y,z)e^{\mathrm{i}\omega t}$. Then we get

$$-\mathrm{i}\omega(\hat{u},\hat{v}) = -(\hat{\phi}_x,\hat{\phi}_y) \tag{9.4}$$

$$-\mathrm{i}\omega\hat{w} = -\hat{\phi}_z + \hat{b} \tag{9.5}$$

$$\hat{u}_x + \hat{v}_y + \hat{w}_z = 0 \tag{9.6}$$

$$-\mathrm{i}\omega\hat{b} + \hat{w}N^2 = 0 \tag{9.7}$$

Substituting (9.4) into (9.6) gives

$$\hat{\phi}_{xx} + \hat{\phi}_{yy} = -\mathrm{i}\omega\hat{w}_z.$$

Combining (9.5) and (9.7) gives

$$\mathrm{i}[N^2-\omega^2]\hat{w} = -\omega\hat{\phi}_z.$$

Eliminating $\hat{\phi}$ between these last two equations gives

$$(N^2-\omega^2)(\hat{w}_{xx}+\hat{w}_{yy}) = \omega^2\hat{w}_{zz} \tag{9.8}$$

Recall, this holds also for vertically varying stratification $N^2 = N^2(z)$.

Effect of Earth's Rotation

At this point, we have neglected the effect of Earth's rotation in the momentum equation. It is straightforward to add $f\hat{\mathbf{k}}\times\mathbf{u}$ to the momentum equation and it only takes a little more algebra but one gets a slightly modified version of (9.8)

$$(N^2-\omega^2)(\hat{w}_{xx}+\hat{w}_{yy}) = (\omega^2-f^2)\hat{w}_{zz} \tag{9.9}$$

This can be rewritten as

$$\hat{w}_{zz} - \frac{(N^2-\omega^2)}{(\omega^2-f^2)}(\hat{w}_{xx}+\hat{w}_{yy}) = 0 \tag{9.10}$$

which reduces back to the form (9.8) if rotation is neglected $f=0$.

9.3 Plane Internal Waves: constant stratification N^2

If $N^2 =$ constant, we can plug in a plane wave solution. Here we are only going to work in two-dimensions such that the vector wavenumber $\mathbf{k} = (k,m)$. All of this can be generalized to 3D with $\mathbf{k} = (k,l,m)$. Plugging in a 2D plane wave solution

$$\hat{w} = \tilde{w}e^{\mathrm{i}(kx+mz)}$$

into (9.8) with $\partial_y = 0$, we obtain the dispersion relation

$$(N^2 - \omega^2)(k^2) = \omega^2 m^2 \tag{9.11}$$

or rearranged to be in the familiar form of $\omega(k, m) = \ldots$,

$$\omega^2 = N^2 \frac{k^2}{(k^2 + m^2)} = N^2 \frac{k^2}{|\mathbf{k}|^2} \tag{9.12}$$

or

$$\boxed{\omega^2 = N^2 \cos^2\theta} \tag{9.13}$$

where θ is the angle of the wave with the horizontal, *i.e.,* $\tan\theta = m/k$. This means that the frequency ω does not depend on the wavelength of the wave, but only on the angle at which it propagates in the stratified fluid. Questions:

- *What are the limitations on ω in order to have waves?*
- *Are these waves dispersive or non-dispersive?*

9.3.1 Simple Internal Wave Kinematics

What do these waves look like? First

$$\nabla \cdot \mathbf{u} = 0 \quad \Longrightarrow \quad \mathbf{k} \cdot \hat{\mathbf{u}} = 0 \tag{9.14}$$

Thus the fluid velocity $\mathbf{u}$ is *perpendicular* to the direction of wave propagation $\mathbf{k}$. In sound waves and surface gravity waves, $\mathbf{u}$ is parallel to $\mathbf{k}$. This means that θ can be thought of as the angle of the direction of velocity and the *vertical*.

Thus, in a rotated coordinate system in the direction of $\mathbf{k}$, the $\mathbf{u}$ components are $\mathbf{u} = (u_{\parallel}, u_{\perp},)$, where $u_{\parallel} = 0$. However, the pressure gradient term is in the direction of $\mathbf{k}$, that is parallel to $\mathbf{k}$, *i.e.,*

$$\nabla\phi = i\mathbf{k}\tilde{\phi}e^{\mathrm{i}(kx+mz-\omega t)}$$

The momentum equation in the direction of $\mathbf{k}$ is thus

$$\partial_{\parallel}\tilde{\phi} = \tilde{b}\sin\theta$$

that is pressure perturbations balance buoyancy (density) flucturations.

In the direction *perpendicular* to $\mathbf{k}$, the the momentum equation in the *upward* direction is

$$\frac{\partial u_{\perp}}{\partial t} = b\cos\theta$$

or

$$\mathrm{i}\omega\tilde{u}_{\perp} = \tilde{b}\cos\theta$$

Next because $w = \mathbf{u}\cdot\hat{\mathbf{k}}$ and θ is defined as $\cos\theta = k/(k^2+m^2)$, then $w = u_{\perp}\cos\theta$. The buoyancy equation is then

$$\frac{\partial b}{\partial t} + N^2 u_{\perp}\cos\theta = 0$$

So part of buoyancy fluctuations goes to balance pressure fluctuations and part of buoyancy flucturations goes to balancing velocity fluctuations. Thus, for these linear internal waves, pressure and velocity are not directly coupled - only indirectly through the buoyancy fluctuations.

9.4 Internal Wave Phase and Group Velocity

From other wave systems we have learned that the energy propagates with a velocity associated with $\mathbf{c}_g = \partial\omega/\partial\mathbf{k}$ For the dispersion relationship (9.12),

$$\omega = N\frac{k}{(k^2+m^2)^{1/2}} = \frac{Nk}{|\mathbf{k}|}$$

the phase velocity $\mathbf{c} = \omega/\mathbf{k}$ is

$$\mathbf{c} = \frac{\omega}{|\mathbf{k}|}\frac{\mathbf{k}}{|\mathbf{k}|} = \frac{\omega}{|\mathbf{k}|^2}(k,m) = \frac{Nk}{|\mathbf{k}|^3}(k,m) \tag{9.15}$$

For the group velocity, we can now calculate

$$\begin{aligned}
\mathbf{c}_g = \left(\frac{\partial\omega}{\partial k},\frac{\partial\omega}{\partial m}\right) &= N\left(\frac{1}{(k^2+m^2)^{1/2}} - \frac{-k^2}{(k^2+m^2)^{3/2}}, \frac{-km}{(k^2+m^2)^{3/2}}\right)\\
&= N\left(\frac{(k^2+m^2}{(k^2+m^2)^{3/2}} - \frac{-k^2}{(k^2+m^2)^{3/2}}, \frac{-km}{(k^2+m^2)^{3/2}}\right)\\
&= N\left(\frac{(m^2}{(k^2+m^2)^{3/2}}, \frac{-km}{(k^2+m^2)^{3/2}}\right)
\end{aligned}$$

$$\mathbf{c}_g = \frac{Nm}{|\mathbf{k}|^3}(m,-k). \tag{9.16}$$

This is pretty interesting. The first thing to note is that $\mathbf{c}\cdot\mathbf{c}_g = 0$ (verify it!) which implies that $\mathbf{c}_g$ is perpendicular to the phase velocity. Second. The direction of $\mathbf{c}_g$ is reflected about the x direction as it is the vertical (z) component that changes sign between $\mathbf{c} \propto (k,m)$ and $\mathbf{c}_g \propto (m,-k)$. Furthermore, the ratio of group to phase speed $|\mathbf{c}_g|/|\mathbf{c}| = m/k$ is a function of the angle θ as $\tan\theta = m/k$.

Table 9.1: Summary of internal wave phase ($\mathbf{c}$) and group ($\mathbf{c}_g$) velocity directions depending on k and m (for positive ω).

k	m	$\mathbf{c}$	$\mathbf{c}_g$
+	+	right, up	right, down
+	-	right, down	right, up
-	+	left, up	left down
-	-	left, down	left, up

9.5 Limiting forms $\omega \to N$

If $\omega \to N$, then $\cos\theta \to 1$, the wavenumber $\mathbf{k}$ is in the horizontal direction and the vertical wavenumber $m \to 0$, *i.e.*, $\mathbf{k} = (k, 0)$. Thus, the fluid velocity is in the vertical direction. This is expected given how we derived the buoyancy frequency earlier. The resulting solution is one of vertical columns oscillating in time at the buoyancy frequency. Note that in this case, pressure fluctuations are *zero*, $u_\perp = w$ and w and b are out of phase.

We can think about now the phase velocity $\mathbf{c}$ and the group velocity $\mathbf{c}_g$ when $\omega \to N$. As $\mathbf{k} = (k, 0)$ (*i.e.*, $m = 0$), then

$$\mathbf{c} = \frac{N}{k}(1, 0)$$

and

$$\mathbf{c}_g = \frac{Nm}{k^3}(0, -1) = 0$$

So the energy propagation speed is minimum as the phase speed is maximum! Note that this is the same physics as just adiabatically lifting a parcel of fluid up in a stratified water column and letting it go! (Section 8.1). Internal waves are crazy.

9.6 More fun linear internal wave facts

9.6.1 Nonlinearity in internal waves

The nonlinear term in the momentum equation was ignored because we assumed small amplitude waves. This linearization was not a formal one by expanding out a small parameter, but a hand-waving one. But with the solution, lets examine the nonlinear terms $\mathbf{u} \cdot \nabla \mathbf{u}$. Recalling that $\mathbf{u} = (u_\parallel, u_\perp) = (0, u_\perp$, the nonlinear term can be written in component form

$$\parallel \text{ direction}, \quad u_\parallel \partial_\parallel u_\parallel + u_\perp \partial_\perp u_\parallel = 0 + u_\perp \times 0 = 0 \tag{9.17}$$

$$\perp \text{ direction}, \quad u_\parallel \partial_\parallel u_\perp + u_\perp \partial_\perp u_\perp = 0 \times \partial_\parallel u_\perp + u_\perp \times 0 = 0 \tag{9.18}$$

So the crazy thing is that the nonlinear terms are zero identically! What does this mean? Are there large amplitude limitations on these waves or not?

9.6.2 Vorticity in internal waves

We can also calculate vorticity in the $\mathbf{u} = (u_\parallel, u_\perp)$ reference frame. Vorticity ζ is written as

$$\zeta = \partial_\perp u_\parallel - \partial_\parallel u_\perp = 0 - i|\mathbf{k}|\hat{u}_\perp e^{\mathrm{i}(\dots)} \neq 0.$$

What does this mean? Not strictly potential flow.

9.7 Aside: Interpreting vector phase velocity

Earlier, it was asserted that the vector phase velocity $\mathbf{c} = \omega/\mathbf{k}$, which implies vector division. This was written as

$$\mathbf{c} = \frac{\omega}{\mathbf{k}} = \frac{\omega}{|\mathbf{k}|}\frac{\mathbf{k}}{|\mathbf{k}|}$$

so that $\mathbf{c}$ and $\mathbf{k}$ are in the same direction.

9.7.1 Bad

Some folks have thought to write $\mathbf{c}$ as (in components)

$$\mathbf{c} = (\omega/k, \omega/l, \omega/m) \tag{9.19}$$

This implies then that

$$|\mathbf{c}| = \omega\left(\frac{1}{k^2} + \frac{1}{l^2} + \frac{1}{m^2}\right)^{1/2}$$

but experience tells us that

$$|\mathbf{c}| = \frac{\omega}{|\mathbf{k}|} = \frac{\omega}{(k^2 + l^2 + m^2)^{1/2}} \tag{9.20}$$

and so clearly something is wrong with (9.19). However, you will find this expression in some folks' lectures!

9.7.2 Why

A wave has a vector form argument that goes into the cos of

$$\theta = \mathbf{k}\cdot\mathbf{x} - \omega t$$

which when thinking of a phase speed gets rewritten as

$$\theta = \mathbf{k}\cdot(\mathbf{x} - \mathbf{c}t)$$

Thus instead of vector division think of it as a dot product

$$\omega = \mathbf{k} \cdot \mathbf{c}$$

Using the definition of $\mathbf{c}$ above (9.20) one gets

$$\mathbf{k} \cdot \mathbf{c} = \mathbf{k} \cdot \frac{\omega}{|\mathbf{k}|} \frac{\mathbf{k}}{|\mathbf{k}|} = \frac{\omega(\mathbf{k} \cdot \mathbf{k})}{|\mathbf{k}|^2} = \omega.$$

Thus this definition of vector division or phase velocity is the right one.

9.8 Problems

Non-rotating fluid: Internal Waves

1. What are the limitations on ω in order to have waves? What happens if $\omega > N$?

2. Are linear internal waves dispersive or non-dispersive? why?

3. If a wave is propagating in the (x, y, z) plane, what is the non-rotating dispersion relationship $\omega(k, l, m)$?

4. For non-rotatiing plane internal waves, at what angle of propagation θ are $|\mathbf{c}|$ and $|\mathbf{c}_g|$ the same?

5. Assuming a deep ocean constant $N = 0.002\ \mathrm{rad\,s^{-1}}$, what is the angle relative to the horizontal made by an internal wave with periods:

   (a) 2 h period

   (b) semi-diurnal internal wave period of 12 h?

   (c) diurnal internal wave period 24 h.

6. Consider an non-rotating ocean of infinite depth with a constant buoyancy frequency N and a constant (steady and depth-uniform) horizontal velocity U. It is convenient to use $z = 0$ as the location of the mean sea bed. Now consider that the sea-bed is rippled such that $z_b = h_0 \cos(k_0 x)$ or $\mathrm{Re}\,[h_0 \exp(\mathrm{i} k_0 x)]$, where the amplitude h_0 is small. You could also think of this as a mountain range in atmospheric flow. I think using the 2nd form is easier.

   (a) Transfer the problem into a coordinate system moving with the mean flow U, *i.e.*, $\tilde{x} = x + Ut$ such that

   $$z_b = h_0 \cos(k_0 \tilde{x} - \omega t),$$

   What is ω?

   (b) What is the appropriate vertical velocity boundary condition at $z = z_b$?

   (c) If h_0 is small, how can this be linearized? Remember surface gravity waves!

   (d) What is the internal wave vertical wavenumber m induced by this bathymetry/topography? Both magnitude and sign? How does it depend on N, U, and k_0?

   (e) What limitation is there on U given k_0 and N in order to generate waves?

   (f) What is the vertical direction of $\mathbf{c}_g$?

(g) Using the boundary condition for w_0, write out the plane wave solution for u, w, ϕ, and b?

Internal Waves in a rotating fluid: Useful to consult MCH notes

1. From the linearized Boussinesq equation (top 7.1) but now adding in rotation on an f-plane so that you have $(-fv, +fu)$ in the momentum equation. Now, don't assume solutions of $e^{\mathrm{i}\omega t}$ but instead manipulate the equations to get a "wave-like" equation for w that looks like
$$\frac{\partial^2}{\partial t^2}\left(\nabla^2 w\right) + f^2 \frac{\partial^2 w}{\partial z^2} + N^2 \nabla_{\mathrm{h}}^2 w = 0 \tag{9.21}$$
where ∇_{h} is the horizontal Laplacian operator.

2. Plugging in a plane wave solution $e^{\mathrm{i}(kx+ly+mz-\omega t)}$ What is the resulting dispersion relationship?

3. What additional limitation does this imply for ω?

Chapter 10

Linear Internal Waves: Energy Conservation

10.1 Local Energy Conservation

REDO THIS PART STARTING FROM PRIMITIVE EQUATIONS! Earlier, we derived an energy conservation equation for the linearized perfect fluid in the Boussinesq approximation which is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}+\frac{1}{2}N^2(\Delta z)^2\right)=-\nabla\cdot(\mathbf{u}\phi) \tag{10.1}$$

For the linearized Boussinesq equations, the potential energy term can be transformed from

$$\frac{1}{2}N^2(\Delta z)^2\rightarrow\frac{b^2}{2N^2(z)}.$$

How does this work? Consider a fluid at $z=z_0$ with buoyancy $b_0(z_0)$ that is displaced a distance Δz from its rest position. This fluid has a buoyancy (*i.e.,* temperature) departure from its new surroundings is

$$b\equiv b_0(z_0)-b_0(z_0+\Delta z)\approx-\frac{\mathrm{d}b_0}{\mathrm{d}z}\Delta z=-N^2\Delta z$$

Thus,

$$\frac{b^2}{2N^2}=\frac{1}{2N^2}(-N^2\Delta z)^2=\frac{N^2}{2}(\Delta z)^2$$

Therefore, energy conservation can be rewritten in the convenient form of

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}+\frac{b^2}{2N^2(z)}\right)=-\nabla\cdot(\mathbf{u}\phi) \tag{10.2}$$

Question: Why is it convenient?

10.2 Linear Internal Wave Kinematics

Now assume N^2 is constant and compute the energy flux in the plane wave. We already know that this energy flux must be perpendicular to the direction of wave propagation. Again assume that $l = 0$, we take

$$\begin{aligned} u_\perp &= \mathrm{Re}\left\{\hat{u}_\perp e^{\mathrm{i}(kx+mz-\omega t)}\right\} \\ b &= \mathrm{Re}\left\{\hat{b} e^{\mathrm{i}(kx+mz-\omega t)}\right\} \\ \phi &= \mathrm{Re}\left\{\hat{\phi} e^{\mathrm{i}(kx+mz-\omega t)}\right\} \end{aligned}$$

Plugging this into the momentum and buoyancy equation yields,

$$\begin{aligned} -\mathrm{i}\omega\hat{u}_\perp &= \hat{b}\cos\theta \\ 0 &= -\mathrm{i}|\mathbf{k}|\hat{\phi} + \hat{b}\sin\theta \\ -\mathrm{i}\omega\hat{b} + N^2\cos\theta\hat{u}_\perp &= 0 \end{aligned}$$

which gives

$$\hat{u}_\perp = \frac{\mathrm{i}\hat{b}\cos\theta}{\omega} \tag{10.3}$$

$$\hat{\phi} = -\frac{\mathrm{i}\hat{b}}{(k^2+m^2)^{\frac{1}{2}}}\sin\theta. \tag{10.4}$$

Note that both of these depend upon $\hat{b}$.

10.3 Average Internal Wave Kinetic and Potential Energy

With the local kinetic energy density $\mathrm{ke} = (1/2)\rho_0\mathbf{u}\cdot\mathbf{u}$, the average kinetic energy density is

$$\langle \mathrm{ke}\rangle = (1/2)\rho_0\langle\mathbf{u}\cdot\mathbf{u}\rangle = \frac{1}{2}\rho_0\langle u_\perp^2\rangle = \frac{1}{4}\rho_0|\hat{u}_\perp|^2. \tag{10.5}$$

using the standard relationship that $\langle a^2\rangle = (1/2)|\hat{a}|^2$. Similarly the average potential energy $\langle \mathrm{pe}\rangle$ is

$$\langle \mathrm{pe}\rangle = \rho_0\left\langle\frac{1}{2}\frac{b^2}{N^2}\right\rangle = \rho_0\frac{|\hat{b}|^2}{4N^2}.$$

Now using the relation above

$$|\hat{u}_\perp|^2 = \frac{|\hat{b}|^2 \cos^2\theta}{\omega^2}$$

then the kinetic energy can be re-written as

$$\langle \mathrm{ke} \rangle = \frac{1}{4}\rho_0 \frac{|\hat{b}|^2 \cos^2\theta}{\omega^2} \tag{10.6}$$

and by the dispersion relation $\omega^2 = N^2 \cos^2\theta$, it can be shown that the mean kinetic and potential energy are equal. Therefore the total mean energy density $\mathcal{E}$ is

$$\mathcal{E} = \langle \mathrm{ke} \rangle + \langle \mathrm{pe} \rangle = \rho_0 \frac{1}{2}\frac{|\hat{b}|^2}{N^2}$$

This has analogies to the energy of a surface gravity wave $E = (1/2)\rho_0 g a^2$. Consider that in the internal wave isopycnals (isotherms) displace an amplitude a_{IW} then by the relationship at top

$$\frac{b^2}{2N^2} = \frac{N^2}{2}(\Delta z)^2$$

then

$$\frac{|\hat{b}|^2}{N^2} = \frac{N^2}{2}(a_{\mathrm{IW}})^2$$

so that

$$\mathcal{E} = \rho_0 \frac{N^2}{2}(a_{\mathrm{IW}})^2$$

which is similar to the form for surface gravity waves. Note that $\mathcal{E}$ has units of $\mathrm{J\,m^{-3}}$ in contrast to surface gravity wave E which has units of $\mathrm{J\,m^{-2}}$. (Recall why this is!)

10.4 Internal Wave Energy Flux

We now calculate the mean internal wave energy flux

$$\mathbf{F} = \rho_0 \langle \mathbf{u}\phi \rangle = \rho_0(\langle u_\parallel \phi \rangle, \langle u_\perp \phi \rangle)) = (0, \langle u_\perp \phi \rangle) \tag{10.7}$$

because $u_\parallel = 0$. Thus, the mean energy flux is in the direction $\perp$ to $\mathbf{k}$. We can evaluate this component with (10.3) and (10.4) as

$$F_\perp = -\rho_0 \frac{1}{2}\frac{|\hat{b}|^2}{\omega(k^2+m^2)^{1/2}} \sin\theta\cos\theta. \tag{10.8}$$

Thus if the internal wave is propagating upward ($\theta > 0$), the energy flux is in the *downward* perpendicular direction. Similarly if the wave is propagating downward ($\theta < 0$) then $F_\perp > 0$. This follows what we learned in the last lecture about $\mathbf{c}_g$ being perpendicular to $\mathbf{k}$, and opposite signed around the vertical.

The energy flux propagates at the speed

$$\frac{F_\perp}{\mathcal{E}} = \frac{-\rho_0 \frac{1}{2} \frac{|\hat{b}|^2}{\omega(k^2+m^2)^{1/2}} \sin\theta\cos\theta}{\rho_0 \frac{1}{2} \frac{|\hat{b}|^2}{N^2}} = \frac{N^2 \sin\theta\cos\theta}{\omega(k^2+m^2)^{\frac{1}{2}}} \tag{10.9}$$

which is the magnitude of the group velocity $|\mathbf{c}_g|$ where $\mathbf{c}_g$ is defined in (9.16). In full vector form $\mathbf{F} = \mathcal{E}\mathbf{c}_g$ where

$$\mathbf{c}_g = \frac{\partial\omega}{\partial\mathbf{k}}.$$

It is left to the problems to demonstrate this.

10.5 Problems

1. Units of internal wave energy flux.

   (a) What are the units of $F_\perp$?

   (b) What are the units of $\int_{-h}^{0} \mathbf{F} \cdot \hat{\mathbf{k}}\, dz$, where $\hat{\mathbf{k}}$ is the unit vector in the horizontal direction?

2. For a linear internal wave propagating in the $+x$ direction ($\theta = 0$). What is the magnitude and direction of the wave energy flux? Can you reason this from the physical definition of energy flux $\langle \mathbf{u}\phi \rangle$?

3. For a linear internal wave propagating in the $+z$ direction ($\theta = \pi/2$). What is the magnitude and direction of the wave energy flux? Why? Can you reason this from the physical definition of energy flux $\langle \mathbf{u}\phi \rangle$?

4. From the expression for $\mathbf{F} = (F_\parallel, F_\perp)$ and $\mathcal{E}$, demonstrate that the $\mathbf{c}_g$ that satisfies $\mathbf{F} = \mathcal{E}\mathbf{c}_g$ also satisfies $\mathbf{c}_g = \partial\omega/\partial\mathbf{k}$.

5. For question #6 in Chapter 9, calculate the vertical energy flux radiating away from the wave bottom bathymetry.

6. For Chapter 9 Problem Set, part II: you derived the internal wave dispersion relation on an f-plane. Continuing on with the rotating internal wave case propagating in the x-z plane, *i.e.,* $\mathbf{k} = (k, 0, m)$ or the y-direction wavenumber $l = 0$, with velocity components (u, v, w)

   (a) Derive relationships between $\hat{u}$, $\hat{b}$, $\hat{\phi}$, $\hat{w}$ and $\hat{v}$ - note that with rotation v is not zero - following Section 9.2.

   (b) At the latitude of San Diego CA for a semi-diurnal IW propagating at $\theta = 45°$ to the horizontal, what is the ratio of $\hat{v}/\hat{u}$?

   (c) Does including rotation affect the Boussinesq fluid energy equation (see Chapter 8 and Eqs. 10.1 and 10.2)? Reason why or why not.

   (d) Following Section 10.4, calculate the internal wave energy flux, with rotation included.

   (e) What group velocity does this energy flux imply? Is it consistent with the dispersion relationship you derived in Chapter 9 Problem Set, part II?

Chapter 11

Linear Internal Waves: Normal Modes

11.1 Internal Wave Reflection from a flat bottom

So what happens when internal waves propagate into a flat horizontal boundary? We've seen for acoustic waves that their reflection is *specular*, where the angle of incidence and reflection are the same.

At a flat horizontal boundary at $z = 0$, the boundary conditions $w = 0$ must be satisfied. As with acoustic waves, we write the incident and reflected IW vertical velocity as $w = w_I + w_R$, where

$$w_I = \hat{w}_I e^{\mathrm{i}(k_I x + m_I z + \omega_I t)} \tag{11.1}$$

$$w_R = \hat{w}_R e^{\mathrm{i}(k_R x + m_R z + \omega_R t)}. \tag{11.2}$$

In order for $w_I + w_R = 0$ at $z = 0$, the following conditions must be met: (1) $\omega_R = \omega_I$, (2) $k_R = k_I$, and (3) $\hat{w}_I = \hat{w}_R$. This leaves us only to figure out the reflected vertical wavenumber m_R. Well as with acoustic waves, we use the dispersion relationship $\omega^2 = N^2 \cos^2\theta$, where θ is the angle of $\mathbf{k}$ to the horizontal. As incident and reflected k and ω are the same, this implies that $m_R = -m_I$. This implies that $\theta_R = \theta_I$ and that internal wave reflection off of a flat horizontal boundary is also *specular*.

Note, unlike acoustic waves, specular reflection only occurs for horizontal flat boundaries. For sloping boundaries things are different, which will be discussed later (Chapter 12).

11.2 Vertical velocity equation and boundary conditions

From the problem set in Chapter 7, you derived a "wave-like" equation for the vertical velocity. That equation for the case of no-rotation is

$$\frac{\partial}{\partial t}\left(\nabla^2 w\right) + N^2(z)\nabla_{\mathrm{h}}^2 w = 0 \tag{11.3}$$

where ∇_{h} is the horizontal Laplacian operator.

Up to now, we did not really consider boundary conditions. We just allowed for plane wave solutions that extended out to infinity in all directions. Well, the ocean has boundaries. Lets say that the vertical velocity $w = 0$ at the surface $z = 0$ and at the bed $z = -h$. This implies that the upper surface of the ocean is a rigid surface. *Do you think this is ok?*

How do we solve (11.3) and get a dispersion relationship? Assume the vertical velocity equation (11.3) has a solution that looks like

$$w(x, z, t) = \hat{w}(z)e^{\mathrm{i}(kx-\omega t)} \quad ,$$

and plug into (11.3) and one gets for $(-h < z < 0)$,

$$\hat{w}_{zz} + \frac{(N^2(z) - \omega^2)}{\omega^2}k^2\hat{w} = 0 \tag{11.4}$$

which, is essentially a restatement of (9.8). We could also write this in the form

$$\hat{w}_{zz} + R^2\hat{w} = 0$$

where

$$R^2 = \frac{(N^2(z) - \omega^2)}{\omega^2}k^2.$$

11.3 Constant stratification N^2

11.3.1 Solution for vertical velocity

Let us now continue to examine the case of constant stratification N^2. This makes (11.4) a constant coefficient 2nd order ODE harmonic oscillator ODE. With the boundary conditions ($\hat{w} = 0$, at $z = 0, -h$), it is straightforward to see that the solution for $\hat{w}$ goes like

$$\hat{w} = \sin\left[\frac{n\pi}{h}(z + h)\right] \tag{11.5}$$

where n is an integer $n = 1, 2, 3, \ldots$. The $\hat{w}$ solution (11.5) matches the boundary conditions. It also matches the ODE (11.4) when

$$(N^2 - \omega^2)k^2 = m^2\omega^2$$
$$m^2 \equiv \frac{n^2\pi^2}{h^2}$$

which is the dispersion relation for internal waves with vertical wavenumber m. Another way of saying this is that with the boundary conditions (11.4) becomes an eigenvalue problem for the vertical wavenumber m which only have solutions for particular m satisfying $m^2 = n^2\pi^2/h^2$. Or, restated, solutions of (11.4) with the boundary conditions exist only for ω and k satisfying

$$\omega^2 = \frac{N^2k^2}{k^2 + \frac{n^2\pi^2}{h^2}} \quad , \quad n = 1, 2, 3, \ldots \tag{11.6}$$

These $\hat{w}(z)$ solutions are shown in Fig. 11.1 and a relevant property of theirs is that they have $n - 1$ zero-crossings.

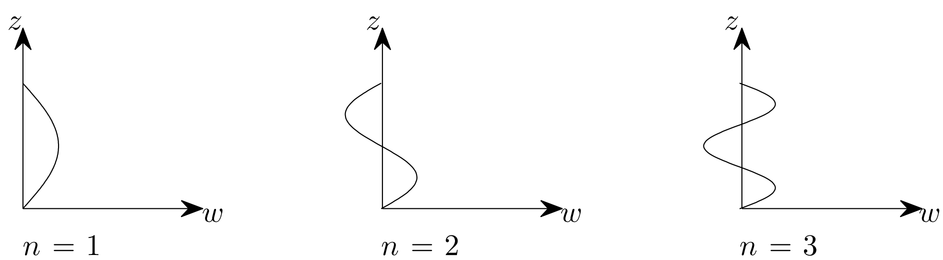


Figure 11.1: Modal structure of vertical velocity w as a function of depth for (left) $n = 1$ to (right) $n = 3$.

Restating, the solutions for $\hat{w}$ are

$$\hat{w}(z) = A \sin\left(\frac{n\pi}{h}(z + h)\right)$$

Where A is an arbitrary constant. Thus the corresponding solution to our equations of motion is

$$w(x, z, t) = \sin\left(\frac{n\pi}{h}(z + h)\right)\left[Ae^{\mathrm{i}(kx - \omega(k,n)t)} + Be^{\mathrm{i}(kx + \omega(k,n)t)}\right] \tag{11.7}$$

where $\omega(k, n)$ is the positive root of (11.6). Since our equations are *linear*, the principle of superposition applies. Thus

$$w(x, z, t) = \int_{-\infty}^{\infty} \mathrm{d}k \sum_{n=1}^{\infty} \left\{A_{nk}e^{-\mathrm{i}\omega_{nk}t} + B_{nk}e^{\mathrm{i}\omega_{nk}t}\right\} \varphi_{nk}(x, z) \tag{11.8}$$

is also a solution, where

$$\varphi_{nk} \equiv \sin\left(\frac{n\pi}{H}(z + H)\right) e^{\mathrm{i}kx} \quad , \quad \omega_{nk} \equiv \omega(k, n) \tag{11.9}$$

and each mode n propagates at it's own frequency and speed.

11.3.2 Remarks:

1. Recall the dispersion relation for internal waves in an *unbounded* fluid (still assuming N^2 =constant),

$$\omega^2 = \frac{N^2 k^2}{k^2 + m^2} \tag{11.10}$$

By comparing (11.6) and (11.10), we see that the presence of boundaries *quantizes* m, which then takes only the values $m = n\pi/h$. This is a typical property of eigenvalue problems, such as the vibrating string of length L (Chapter 1).

2. With $m = n\pi/h$ and $z' \equiv z + h$, the basic solution

$$w = A\sin(mz')e^{\mathrm{i}(kx-\omega t)}$$

is a *standing wave* in the vertical and (as before) can be viewed as the sum of progressive plane waves,

$$w = \frac{A}{2i}\left[e^{\mathrm{i}(kx+mz'-\omega t)} - e^{\mathrm{i}(kx-mz'-\omega t)}\right]$$

propagating in the directions $(k, \pm m)$ with amplitudes and phases arranged to produce perfect cancellation at the boundaries $z = (0, h)$. So we see again that the principle applies of a standing wave being the sum of opposing propagating progressive waves.

3. The $\{\varphi_{nk}\}$ is a set of orthogonal functions:

$$\begin{aligned}
&\int_{-h}^{0} \mathrm{d}z \int_{-\infty}^{\infty} \mathrm{d}x\, [\varphi_{nk}\varphi^*_{n'k'}] \\
&= \left\{\int_{-h}^{0} \mathrm{d}z \sin\left(\frac{n\pi}{h}(z+h)\right)\sin\left(\frac{n'\pi}{h}(z+h)\right)\right\}\cdot\left\{\int_{-\infty}^{\infty} e^{\mathrm{i}(k-k')x}\,\mathrm{d}x\right\} \\
&= \left\{\frac{h}{\pi}\frac{\pi}{z}\delta_{nn'}\right\}\cdot\{2\pi\delta(k-k')\} \\
&= h\pi\,\delta_{nn'}\,\delta(k-k')
\end{aligned}$$

Where $\delta_{nn'}$ is the Kronecker delta function and $\delta(k-k')$ is the Dirac delta function. This orthogonality property is of course very useful in calculating the coefficients of each mode in (11.8). See, for example, Figure 11.2.

4. The $\{\varphi_{nk}\}$ are a *complete* set of basis functions. Thus (11.8) is the *general solution* of the problem.

11.3.3 Buoyancy and Horizontal Velocity Fluctuations

Recall that the linearized buoyancy equation is

$$\frac{\partial b}{\partial t} + wN^2 = 0$$

, and with the solution for $w(x, y, z, t)$ of (11.7), we can write the solution for

$$b(x, z, t) = \frac{AN^2}{\mathrm{i}\omega} \sin\left(\frac{n\pi}{h}(z + h)\right) e^{\mathrm{i}(kx - \omega(k,n)t)} + \frac{BN^2}{\mathrm{i}\omega} \sin\left(\frac{n\pi}{h}(z + h)\right) e^{\mathrm{i}(kx + \omega(k,n)t)} \quad (11.11)$$

with the full spectral solution similar to (11.8). The velocities can be related from the continuity equation $u_x + w_z = 0$ which becomes $\mathrm{i}k\hat{u} = Am\cos(m(z + h))$ and thus

$$u(x, z, t) = \frac{Am}{\mathrm{i}k} \cos\left(\frac{n\pi}{h}(z + h)\right) e^{\mathrm{i}(kx - \omega(k,n)t)} + \frac{Bm}{\mathrm{i}k} \cos\left(\frac{n\pi}{h}(z + h)\right) e^{\mathrm{i}(kx + \omega(k,n)t)} \quad (11.12)$$

This means that if you know the mode-1 u fluctuations, you can infer the mode-1 buoyancy or temperature fluctuations, or vice-versa. This is very similar to how with linear surface gravity waves, you can use velocity to infer pressure, and vice versa (recall problem set from Chapter 2). *What about solutions for $\phi = p/\rho$?*

11.3.4 Normal Mode Phase and Group Velocity

With the normal-model internal wave dispersion relationship,

$$\omega^2 = \frac{N^2 k^2}{k^2 + \frac{n^2\pi^2}{h^2}} \quad , \quad n = 1, 2, 3, \ldots$$

Note that the denominator can be re-written as

$$h^{-2}\left((kh)^2 + n^2\pi^2\right)$$

so that

$$\omega^2 = \frac{N^2(kh)^2}{(kh)^2 + (n\pi)^2},$$

where now we see the re-appearance of our old friend from surface gravity waves - namely kh the non-dimensionalized depth parameter. We can also write,

$$\omega = \frac{N(kh)}{((kh)^2 + (n\pi)^2)^{1/2}}$$

Now, we can calculate the phase and group velocity in the $+x$ direction

$$c = \frac{\omega}{k} = \frac{Nh}{((kh)^2 + (n\pi)^2)^{1/2}}.$$

and the group velocity in the $+x$ direction as

$$
\begin{aligned}
c_g = \frac{\partial \omega}{\partial k} =& \frac{Nh}{\left((kh)^2 + (n\pi)^2\right)^{1/2}} + (Nkh)\frac{-1}{2}(2kh^2)\left((kh)^2 + (n\pi)^2\right)^{-3/2} \\
=& (Nh)\frac{(kh)^2 + (n\pi)^2 - (kh)^2}{\left((kh)^2 + (n\pi)^2\right)^{3/2}} \\
=& \frac{Nh(n\pi)^2}{\left((kh)^2 + (n\pi)^2\right)^{3/2}}
\end{aligned}
$$

Do you see the analogy with the mode phase and group velocity and the vector IW phase and group velocity? Are these waves dispersive or non-dispersive? Recall that $m = n\pi/h$.

11.3.5 The shallow water limit: $kh \ll 1$

Just like there is a shallow water limit for surface gravity waves, there is a shallow water limit for modal internal waves. If we re-write

$$
\left((kh)^2 + (n\pi)^2\right)^{1/2} = (n\pi)\left(1 + (kh)^2/(n\pi)^2\right)^{1/2}
$$

and

$$
\omega = \frac{Nhk}{n\pi(1 + (kh)^2/(n\pi)^2))^{1/2}}.
$$

If we let $(kh)/(n\pi) \ll 1$ then $\omega = Nhk/(n\pi)$ and the phase speed c becomes

$$
c = \frac{\omega}{k} = \frac{Nh}{n\pi}.
$$

This is pretty interesting as now the phase speed is non-dispersive. All frequencies or k propagate at the same speed. This is analogous to shallow water surface gravity waves. Let us examine the group velocity in same limit $(kh)/(n\pi) \ll 1$

$$
c_g = \frac{\partial \omega}{\partial k} = \frac{Nh(n\pi)^2}{\left((kh)^2 + (n\pi)^2\right)^{3/2}} \approx \frac{Nh}{n\pi}
$$

and as expected for non-dispersive waves the phase and group velocity are the same $c = c_g$.

11.3.6 Energetics

In Chapter 10, we derived a local energetics equation (10.2) that was the sum of kinetic and potential energy balancing an energy flux divergence

$$
\rho_0 \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{b^2}{2N^2(z)}\right) = -\rho_0 \nabla\cdot(\mathbf{u}\phi) \qquad \text{(10.2 repeated)}
$$

where the local energy $\mathcal{E}(z)$ was (units of J/m^3).

$$\mathcal{E}(z) = \langle \text{ke} \rangle + \langle \text{pe} \rangle = \rho_0 \frac{1}{2} \frac{|\hat{b}(z)|^2}{N^2}$$

As with surface gravity waves, this can be vertically integrated to

$$E = \int_{-h}^{0} \mathcal{E}(z) \, \mathrm{d}z = \frac{\rho_0}{2N^2} \hat{b}_0^2 \int_{-h}^{0} \sin^2 \left(\frac{n\pi}{h}(z+h) \right) \, \mathrm{d}z. \tag{11.13}$$

What units does E have? Similarly the energy flux can be vertically integrated to

$$F = \int_{-h}^{0} \langle u\phi \rangle \, \mathrm{d}z \tag{11.14}$$

Question: What units does this F have? Question: can you evaluate F and show that $F = Ec_g$?

11.4 Varying Stratification: Non-constant $N^2(z)$

In the ocean, $N^2(z)$ is very non-uniform. How much of the above remains true in the case of non-uniform $N^2(z)$? Answer: virtually all.

To consider non-uniform $N^2(z)$, we must return to (11.15), rewritten as

$$\frac{\mathrm{d}^2 \hat{w}}{\mathrm{d}z^2} + \lambda r(z) \hat{w} = 0, \quad \hat{w} = 0 \quad \text{at} \quad z = 0, -h \tag{11.15}$$

where now,

$$r(z) \equiv \frac{N^2(z) - \omega^2}{\omega^2} \quad , \quad \lambda \equiv k^2 \tag{11.16}$$

Again we regard ω (i.e. $r(z)$) as given, and try to find k (i.e. λ) such that (11.15) is satisfied. Obviously $\hat{w} \equiv 0$ is always a solution. However, nonzero $\hat{w}$ exist but only for very special values of λ. These are called *eigenvalues* and (11.15) is called an *eigenvalue problem.*

11.4.1 Sturm-Liouville Eigenfunction Theory - Light

The general theory of eigenvalue problems is called Sturm-Liouville theory. It considers equations of the general form

$$\frac{\mathrm{d}}{\mathrm{d}z} \left[p(z) \frac{\mathrm{d}w}{\mathrm{d}z} \right] + [q(z) + \lambda r(z)] \, w = 0$$

of which (11.15) is a special case (although $r(z) > 0$ is often assumed in texts). See Morse & Feshbach pp. 719-729.] Here $p(x) = 1$ and $q(x) = 0$.

11.4.2 Properties of internal wave modal structure

Let $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ be the eigenvalues of (11.15) and $\{\hat{w}_1(z), \hat{w}_2(z), \hat{w}_3(z), \ldots\}$ the corresponding eigenfunctions. Then we can show the following:

1. *Solutions with $\lambda > 0$ (k real, as we require for waves) occur only for $\omega < N_{MAX}$.* Let $\hat{w}(z)$ and λ be any eigenfunction-eigenvalue pair. Then multiply

   $$\hat{w}_{zz} + \lambda r(z)\hat{w} = 0$$

   by $\hat{w}$, and integrate from $z = -h$ to $z = 0$, using the boundary conditions. The result is

   $$\lambda = \frac{\int_{-h}^{0} (\hat{w}_z)^2 \, \mathrm{d}z}{\int_{-h}^{0} r(z)\hat{w}^2 \, \mathrm{d}z}$$

   Since we require $\lambda > 0$ then

   $$\int_{-h}^{0} r(z)\hat{w}^2 \, \mathrm{d}z > 0,$$

   and thus $r(z)$ must be positive in some part of its domain. Thus $N^2(z) - \omega^2 > 0$ somewhere in the vertical. If this is not the case what happens to the eigenvalue λ and what form does the solution take?

2. The solution $\hat{w}(z)$ is non-oscillatory where $r(z) < 0$. To show this, integrate

   $$\hat{w}_{zz} + \lambda r(z)\hat{w} = 0$$

   between two supposed zeroes, between which $r(z) < 0$, and demonstrate a contradiction.

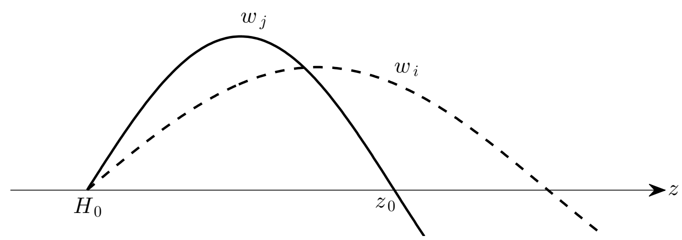
3. Where $r(z) > 0$, the zeroes of w_i are closer together than the zeroes of w_j provided $\lambda_i > \lambda_j$. To show this, multiply

   $$\begin{aligned} \hat{w}_i \quad &\text{by} \quad \hat{w}_j'' + r(z)\lambda_j \hat{w}_j = 0 \\ \hat{w}_j \quad &\text{by} \quad \hat{w}_i'' + r(z)\lambda_i \hat{w}_i = 0 \end{aligned}$$

   and subtract to get

   $$(\hat{w}_i\hat{w}_j' - \hat{w}_j\hat{w}_i')' + (\lambda_j - \lambda_i)r(z)\hat{w}_i\hat{w}_j = 0$$

   and integrate from $z = -h$ to z_0, the first zero crossing of $\hat{w}_j$.
   We will suppose that the first zero of $\hat{w}_i$ occurs at $z > z_0$, and demonstrate a contradiction.



Then

$$\underbrace{\hat{w}_i(z_0)}_{+}\,\underbrace{\hat{w}_j'(z_0)}_{-} = \underbrace{(\lambda_i - \lambda_j)}_{+}\int_{-h}^{z_0} \underbrace{r(z)\hat{w}_i\hat{w}_j}_{+}\,\mathrm{d}z$$

which is a contradiction. It follows from all this that there is a lowest eigenvalue corresponding to no zero crossings ($\hat{w}$ all of one sign).

4. The $\{\hat{w}_i(z)\}$ are orthogonal. To see this, integrate

$$(\hat{w}_i\hat{w}_j' - \hat{w}_j\hat{w}_i')' + (\lambda_j - \lambda_i)r(z)\hat{w}_i\hat{w}_j = 0$$

between $-h$ and 0. It follows that

$$(\lambda_j - \lambda_i)\int_{-h}^{0} r(z)\hat{w}_i\hat{w}_j = 0$$

Thus if $\lambda_j \neq \lambda_i$,

$$\int_{-h}^{0} r(z)\hat{w}_i(z)\hat{w}_j(z) = 0$$

and the functions are orthogonal with respect to the weight $r(z)$.

The orthogonality property is useful for representing arbitrary functions as sums of the $\{\hat{w}_i\}$. Let $f(z)$ be an arbitrary function. Then if constants c_i can be found such that

$$f(z)\sum_{i=1}^{\infty} c_i\hat{w}_i(z)$$

it *must* be true that

$$c_i = \frac{\int r(z)f\hat{w}_i\,\mathrm{d}z}{\int r(z)\hat{w}_i^2\,\mathrm{d}z}.$$

This is totally analogous to Fourier series and determining Fourier coefficients.

5. The $\{\hat{w}_i(z)\}$ are complete, in the sense that given any $f(z)$, such c_i's always exist to make the first equation true. (proof not given here.)

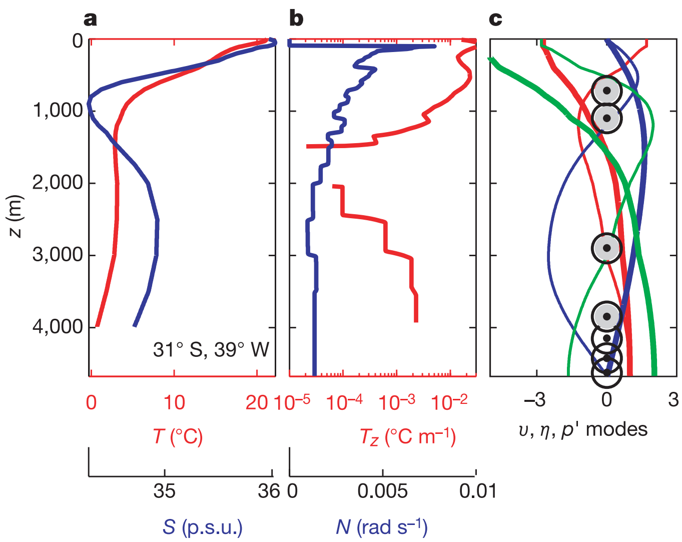


Figure 1 Hydrographic profiles, mode structure and instrument depths for a typical mooring. **a**, Profiles[21] of climatological temperature (red) and salinity (blue) at the mooring location (31° S, 39° W). **b**, Temperature gradient T_z (red) and buoyancy frequency $N(z)$ (blue) computed from **a**. **c**, Thick and thin lines show respectively first and second vertical normal modes of velocity **u** (red), vertical displacement η (blue), and pressure anomaly p' (green). Circles indicate instrument depths. In this and all calculations, temperature measurements where $T_z < 3\times10^{-5}$ °C m^{-1}, or where T increases towards the sea floor, are removed from the calculation (bottom three records here). Briefly, flux is computed as follows (see Methods for details): η is inferred from band-pass-filtered temperature records and climatological profiles (**a**, **b**). A least-squares solution is obtained for the amplitude of the two longest-wavelength normal modes (**c**) in terms of the discrete-depth moored measurements. Depth-integration yields p' in terms of η. The correlation between velocity and p' equals the energy flux.

Figure 11.2: Example of realistic modal decomposition: From Alford Nature (2003)

11.5 Problems

1. In the internal wave shallow water limit ($(kh)/(n\pi) \ll 1$), for the vertical momentum equation, *i.e.*,
$$\frac{\partial w}{\partial t} = -\rho_0^{-1}\frac{\partial p}{\partial z} - \frac{\rho' g}{\rho_0}$$
what is the dominant dynamical balance? Does this have a name?

2. In the problem set for Chapter 7, you derived a vertical velocity equation that included rotation (9.21).
$$\frac{\partial}{\partial t}\left(\nabla^2 w\right) + f^2\frac{\partial^2 w}{\partial z^2} + N^2\nabla_{\mathrm{h}}^2 w = 0,$$
where ∇_{h} is the horizontal Laplacian operator. Here, you are going to repeat the derivation in Sections 9.1–9.3 for constant stratification N^2 and using the boundary conditions of $w = 0$ at $z = 0$ and $z- = h$,

   (a) Plug in a solution $w = \hat{w}(z)e^{\mathrm{i}(kx-\omega t)}$ and derive an ODE for the vertical velocity

   (b) apply the boundary conditions and solve for the vertical modal structure of $\hat{w}(z)$ and the dispersion relationship $\omega^2 = \ldots$.

   (c) For a mode $n = 1$, single plane wave propagating in $+x$ with frequency ω, what is the the solution for $b(x, z, t)$, $u(x, z, t)$?

   (d) What is the solution for $v(x, z, t)$? Why is $v \neq 0$?

   (e) What is the phase and group velocity with rotation?

3. For a mode-1 IW propagating in $+x$ direction at frequency ω_0 with vertical velocity amplitude w_0, using the result also for $b(x, z, t)$, derive an expression for $\phi(x, z, t)$.

4. (EXTRA CREDIT) Show that $F = Ec_g$ for normal mode internal waves where F is defined in (11.14) and E is defined in (11.13) and $c_g = \partial\omega/\partial k$.

5. (EXTRA CREDIT) For an variable but positive $N^2(z)$ profile where the semi-diurnal radian frequency was always $< N$ everywhere, for a plane semi-diurnal internal wave in the ocean, sketch out how you might calculate the depth-integrated horizontal energy flux (units $\mathrm{W\,m^{-1}}$),
$$\int_{-h}^{0} \langle up \rangle \,\mathrm{d}z.$$

Chapter 12

Internal Wave Reflection on a Slope

12.1 Streamfunction definition of IW velocities

In some situations, such as IW reflection on a slope, it is convenient to work with an internal wave streamfunction $\psi(x, z, t)$ defined in the classical manner where for a right-handed coordinate system,

$$\frac{\partial \psi}{\partial z} = u, \qquad \frac{\partial \psi}{\partial x} = -w$$

Aside: Why is this decomposition ok? Using the plane wave solution for $w(x, y, z, t)$ it is easy to show that for each plane wave component, the streamfunction looks like

$$\psi(x, y, z, t) = \frac{\hat{w}}{\mathrm{i}k} e^{\mathrm{i}(kx+mz-\omega t)} = \hat{\psi} e^{\mathrm{i}(kx+mz-\omega t)} \tag{12.1}$$

12.2 Definition of bottom boundary

Suppose the bottom slopes with an ange β such that the bed is located at $z = x \tan \beta$. We can define a unit vector parallel to the slope and perpendicalar to the slope as

$$\hat{\mathbf{n}}_{\parallel} = \hat{\mathbf{i}} \cos \beta + \hat{\mathbf{k}} \sin \beta$$
$$\hat{\mathbf{n}}_{\perp} = -\hat{\mathbf{i}} \sin \beta + \hat{\mathbf{k}} \cos \beta$$

The bottom boundary condition here is that $\mathbf{u} \cdot \hat{\mathbf{n}}_{\perp} = 0$, which can also be written as

$$\frac{\partial \psi}{\partial x_{\parallel}} = 0 \tag{12.2}$$

which implies that ψ is a constant on the boundary which can be taken for convenience as $\psi = 0$ on the boundary $z = x \tan \beta$

12.3 Superposition of Incident and Reflected Wave

We can superimpose the streamfunctions for incident and reflected waves, $\psi = \psi_R + \psi_I$ and this must be zero on the boundary. In order for this to work then

$$(k_I + m_I \tan\beta)x - \omega_I t = (k_R + m_R \tan\beta)x - \omega_R t \tag{12.3}$$

which implies that the frequency is conserved upon reflection

$$\omega_R = \omega_I \tag{12.4}$$

and that the component of wavenumber parallel to the slope ($\hat{\mathbf{n}}_\parallel$) is conserved.

$$(k_I + m_I \tan\beta)x = (k_R + m_R \tan\beta)x \tag{12.5}$$

which can be written as

$$\mathbf{k}_R \cdot \hat{\mathbf{n}}_\parallel = \mathbf{k}_I \cdot \hat{\mathbf{n}}_\parallel \tag{12.6}$$

These are general results for the reflection of plane waves. Typically, the reflected wavenumber in the perpincular direction $\hat{\mathbf{n}}_\perp$ is then derived from the dispersion relationship.

In acoustic wave reflection, the dispersion relationship does not depend upon the wave angle only on the wavenumber magnitude. This implied that if ω and $k_\parallel$ was conserved then $k_\perp$ was also conserved. But with internal waves, $\omega^2 = N^2 \cos^\theta$ and so if ω is conserved with reflection then θ must also be conserved. What this means is that $k_\perp$ is not generally conserved - only for flat bottoms.

To recap:

1. ω is conserved

2. $k_\parallel$ is conserved ($\mathbf{k}_R \cdot \hat{\mathbf{n}}_\parallel = \mathbf{k}_I \cdot \hat{\mathbf{n}}_\parallel$)

3. $\theta_R = \theta_I$

What this implies is that for a slope of angle β,

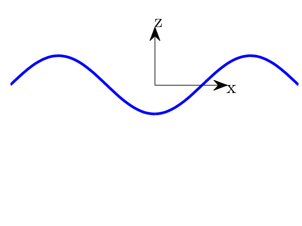
$$|\mathbf{k}_I| \cos(\theta - \beta) = |\mathbf{k}_R| \cos(\theta + \beta) \tag{12.7}$$

Chapter 13

Lecture: Combined Internal and Surface Waves

Now that we have studied surface gravity waves and internal waves separately, we now examine them *jointly*. In particular, in examining the modal structure of internal waves (Chapter 11), we made the assumption that the vertical velocity at the surface was zero, $w = 0$ at $z = 0$ m. This gave us convenient modal solutions. But, is the free surface really fixed for internal waves? How do surface gravity waves influence internal waves? Or how does stratificatino affect surface gravity waves? We will address these questions in a *linear* context.

13.1 Full equations and bounary conditions



With a free surface at $z = \eta(x, y, z)$ and flat (constant) bottom at $z = -h$, the fully nonlinear Boussinesq-approximation pefect fluid equations and boundary conditions are

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla\varphi + b\hat{\mathbf{k}}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} + N^2(z)w = 0$$

with boundary conditions

$$\left.\begin{cases} w = \dfrac{\mathrm{D}\eta}{\mathrm{D}t} \\ p = 0 \end{cases}\right\} \quad \text{at } z = \eta(x, y, t)$$

$$w = 0 \quad \text{at } z = -h$$

Where, we use φ now for density normalized pressure deviations from hydrostatic, *i.e.,* $\varphi = p/\rho_0$, where p is the perturbation pressure. Note that earlier we used $\phi = p/\rho_0$, and the change in notation is explained below. Also, note that here yet again, surface tension has been neglected. And so this analysis explicitly ignores capillary waves as did the surface gravity wave analysis of Chapter 2.

The corresponding *linearized* equations and boundary conditions are:

$$\boxed{\begin{array}{lll} \dfrac{\partial \mathbf{u}}{\partial t} = -\nabla\varphi + b\hat{\mathbf{k}} & \left.\begin{array}{l} w = \dfrac{\partial \eta}{\partial t} \\ \varphi = g\eta \end{array}\right\} & \dfrac{\partial \varphi}{\partial t} = gw \quad \text{at} \quad z = 0 \\ \nabla \cdot \mathbf{u} = 0 & & \\ \dfrac{\partial b}{\partial t} + N^2(z) w = 0 & & w = 0 \quad \text{at} \quad z = -h \end{array}} \qquad \bigstar$$

where the boundary condition $\varphi = g\eta$ at $z = 0$ is derived by Taylor series expansion about $z = 0$, as we've seen previously for surface gravity waves (Chapter 2). By writing total pressure $p = p_0 + \rho_0\varphi$ where p_0 is the hydrostatic pressure, the Taylor series expansion about

$$p = 0 \quad \text{at} \quad z = \eta$$

about $z - 0$ is

$$\begin{aligned} p(z = \eta) = & \quad p(0) + \frac{\partial p}{\partial z}(0)\eta + \frac{\partial^2 p}{\partial z^2}(0)\frac{\eta^2}{2} + \cdots = 0 \\ = & \quad [p_0(0) + \rho_0\varphi(0)] + [-\rho_0 g + \rho_0 \frac{\partial \varphi}{\partial z}(0)]\eta + \cdots = 0 \\ = & \quad \rho_0\varphi(0) - \rho_0 g\eta + \text{ quadratic terms } = 0 \end{aligned}$$

yielding to first order that $\varphi = g\eta$ at $z = 0$. We now concentrate on the system $\bigstar$.

13.2 Simple surface waves: no stratification $N^2 \equiv 0$

This problem was solved in Chapter 2 by assuming irrotational only flow and then finding a solution for the velocity potential ϕ (where $\mathbf{u} = \nabla\phi$) based on $\nabla^2\phi = 0$. This requires that the fluid vorticity be zero. But we've seen from Chapter 9 that vorticity is not zero in internal waves. So, to proceed generally to allow both surface and internal waves, we do not assume irrotational.

With no stratification, the system ★ reduces to

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla\varphi \qquad\qquad \frac{\partial \varphi}{\partial t} = gw \quad \text{at} \quad z=0$$
$$\nabla\cdot\mathbf{u} = 0 \qquad\qquad w = 0 \quad \text{at} \quad z=-H$$

We instead take inspiration from the modal IW solution and plug in a $+x$ propagating wave solution with arbitrary vertical strcuture. Assuming, the flow is unbounded in (x, y), we can seek solutions in the form

$$\mathbf{u} = \hat{\mathbf{u}}(z)e^{\mathrm{i}(kx-\omega t)} \quad , \quad \varphi = \hat{\varphi}(z)e^{\mathrm{i}(kx-\omega t)}$$

where the y-wavenumber $l = 0$ is assumed with no loss in generality. Then:

$$-\mathrm{i}\omega\hat{u} = -\mathrm{i}k\hat{\varphi} \tag{13.1a}$$
$$-\mathrm{i}\omega\hat{v} = 0 \tag{13.1b}$$
$$-\mathrm{i}\omega\hat{w} = -\frac{\partial\hat{\varphi}}{\partial z} \tag{13.1c}$$
$$\mathrm{i}k\hat{u} + \frac{\partial \hat{w}}{\partial z} = 0 \tag{13.1d}$$
$$-\mathrm{i}\omega\hat{\varphi}(0) = g\hat{w}(0) \tag{13.1e}$$
$$\hat{w}(-h) = 0 \tag{13.1f}$$

The first thing this implies is the trivial result that $\hat{v} = 0$.

13.2.1 Equation for vertical velocity

Now adding (13.1a) and (13.1d) yields

$$\frac{ik^2}{\omega}\hat{\varphi} + \frac{\partial \hat{w}}{\partial z} = 0 \tag{13.2}$$

and then taking a vertical derivative and combining with (13.1c) yields an equation for the Fourier transform of the vertical velocity

$$\hat{w}_{zz} = k^2\hat{w}, \tag{13.3}$$

which is a pretty familiar 2nd order constant coefficient ODE. Similarly, the $\hat{w}$ boundary conditions are

$$g\hat{w} = \frac{\omega^2}{k^2}\hat{w}_z \quad \text{at} \quad z=0 \tag{13.4}$$
$$\hat{w} = 0 \quad \text{at} \quad z=-h \tag{13.5}$$

The solution to (13.3) is

$$\hat{w} = Ae^{kz} + Be^{-kz},$$

and more conveniently:

$$\hat{w}(z) = A\cosh kz + B\sinh kz$$

Now think about what would happen if k^2 were *negative*...

13.2.2 Aside:

Recall:

$$\cosh z = \frac{e^z + e^{-z}}{2} \qquad \sinh z = \frac{e^z - e^{-z}}{2}$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\cosh z = \sinh z \quad , \quad \frac{\mathrm{d}}{\mathrm{d}z}\sinh z = \cosh z\right)$$

13.2.3 Application of boundary conditions..

Bottom boundary conditions is $w = 0$ at $z = -h$ yields

$$A\cosh kh - B\sinh kh = 0$$

and thus $A = B\tanh(kh)$. Applying the surface boundary conditions results in

$$gA = \frac{\omega^2}{k^2}kB$$

That when combined $\implies A = B = 0$ unless

$$\boxed{\omega^2 = gk\tanh kh}$$

which is of course the familiar general dispersion relation. As we recalled, this has limiting cases,

I. deep water waves ($kh \to \infty$) $\quad \omega^2 = gk \quad$ (dispersive)

II. shallow water waves ($kh \to 0$) $\quad \omega^2 = (gh)k^2 \quad$ (nondispersive)

with solutions,

$$\begin{aligned}
w &= B\left[\frac{\omega^2}{gk}\cosh kz + \sinh kz\right]e^{\mathrm{i}(kx-\omega t)} \\
&= B[\tanh kh \cdot \cosh kz + \sinh kz]e^{\mathrm{i}(kx-\omega t)} \\
&= B'[\sinh k(h+z)]e^{\mathrm{i}(kx-\omega t)} \\
\varphi &= iB'\frac{\omega}{k}\cosh[k(h+z)]e^{\mathrm{i}(kx-\omega t)} \\
&= B'\frac{\omega}{k}\cosh[k(h+z)]e^{\mathrm{i}(kx-\omega t+\frac{\pi}{2})} \\
u &= B'\cosh[k(h+z)]e^{\mathrm{i}(kx-\omega t+\frac{\pi}{2})}
\end{aligned}$$

Thus, the horizontal velocity, pressure, and surface elevation are in phase, but the vertical velocity *leads* the pressure by $\pi/2$ (or 90°).

13.2.4 Deep and shallow water limiting cases reviewed

I. deep water waves ($kh \to \infty$)

$$w = B'' e^{kz} \cos(kx - \omega t)$$
$$\varphi = B'' \frac{\omega}{k} e^{kz} \cos(kx - \omega t + \tfrac{\pi}{2})$$
$$u = B'' e^{kz} \cos(kx - \omega t + \tfrac{\pi}{2})$$

Particle trajectories are circles.

II. shallow water waves ($kh \to 0$)

$$w = B' kh(1 + \tfrac{z}{h}) \cos(kx - \omega t)$$
$$\varphi = B' \sqrt{gh} \cos(kx - \omega t + \tfrac{\pi}{2})$$
$$u = B' \cos(kx - \omega t + \tfrac{\pi}{2})$$

Particle trajectories are ellipses.

13.2.5 Linkage with Irrotational-based derivation

Recall, that in Chapter 2 we started off *a priori* with irrotational assumption. It turns out that this identical solution is also *irrotational*, as can be seen at once from the equation

$$-\mathrm{i}\omega \hat{\mathbf{u}} = -\nabla \varphi \quad \Longrightarrow \quad \hat{\mathbf{u}} = \nabla \Phi \qquad \Phi = \frac{\varphi}{\mathrm{i}\omega}.$$

The boundary condition $\frac{\partial \varphi}{\partial t} = gw$ can be viewed as a linearization of Bernoulli's equation, which applies to general nonlinear *irrotational* flow. This way of proceeding emphasizes that, for surface waves, the basic equation is *elliptic* and the waves arise from the boundary condition.

13.3 Combined surface and internal waves: the case $N^2 =$ constant

13.3.1 A combined surface and internal wave equation for vertical velocity

In this case, our equations are those in system ★. Again seeking plane wave solutions in the form

$$\varphi = \hat{\varphi}(z) e^{\mathrm{i}(kx - \omega t)}, \quad \text{etc}$$

and we obtain a familiar system of equations

$$-\mathrm{i}\omega\hat{u} = \mathrm{i}k\hat{\varphi} \tag{13.6}$$

$$-\mathrm{i}\omega\hat{w} = -\frac{\partial\hat{\varphi}}{\partial z} + \hat{b} \tag{13.7}$$

$$\mathrm{i}k\hat{u} + \frac{\partial\hat{w}}{\partial z} = 0 \tag{13.8}$$

$$-\mathrm{i}\omega\hat{b} + N^2\hat{w} = 0 \tag{13.9}$$

with boundary conditions

$$-\mathrm{i}\omega\hat{\varphi} = g\hat{w} \quad \text{at} \quad z = 0 \tag{13.10}$$

$$\hat{w} = 0 \quad \text{at} \quad z = -h \tag{13.11}$$

Important: notice that the only difference with the modal solution in Chapter 11 is that the the vertical velocity matches (13.10) instead of the previously assumed $w = 0$ at $z = 0$.

Now, as before, to get an equation for the vertical velocity we combin (13.6) and (13.8) to get

$$\frac{\mathrm{i}k^2}{\omega}\hat{\varphi} + \frac{\partial\hat{w}}{\partial z} = 0$$

and (13.7) and (13.9) to get

$$[N^2 - \omega^2]\hat{w} = \mathrm{i}\omega\frac{\partial\hat{\varphi}}{\partial z}.$$

This can then be combined to get

$$\hat{w}_{zz} + \frac{k^2(N^2 - \omega^2)}{\omega^2}\hat{w} = 0$$

or

$$\hat{w}_{zz} + R^2\hat{w} \tag{13.12}$$

where

$$R^2 = \frac{k^2(N^2 - \omega^2)}{\omega^2}. \tag{13.13}$$

The surface and bottom boundary conditions are:

$$\hat{w} = \frac{\omega^2}{gk^2}\frac{\mathrm{d}\hat{w}}{\mathrm{d}z} \quad \text{at} \quad z = 0 \tag{13.14}$$

$$\hat{w} = 0 \quad \text{at} \quad z = -h \tag{13.15}$$

13.3.2 Solutions in the limit of high and low frequency

There are two possibilities for constant and vertically uniform N^2.

High frequency solutions: $\omega^2 > N^2$

For $\omega^2 > N^2$) then we have $R^2 < 0$. This means that we can't have verticaly oscillating solutions but instead have exponential of the form

$$e^{\pm \tilde{k} z}$$

where

$$\tilde{k}^2 \equiv -R^2.$$

We already knew that internal wave solutions were not possible for $\omega^2 > N^2$, that is that the solutions were evanenscent. The particular solution

$$\hat{w} = B \sinh[\tilde{k}(h+z)]$$

satisfies the equation and bottom boundary condition. The surface boundary condition then implies

$$\omega^2 = gk^2 \frac{\tanh(\tilde{k}h)}{\tilde{k}}$$

which looks similar to the standard surface gravity wave dispersion relationship.

Note: if $N^2 = 0$, then $\tilde{k} = k$ and this case recovers standard surface waves. Also, in the limit that $\omega \gg N$ then $R^2 \to k^2$ and standard surface gravity waves are also recovered.

Low frequency solutions: $\omega^2 < N^2$

When $\omega^2 < N^2$, we expect to have internal wave solutions. This implies that $R^2 > 0$, which gives oscillating solutions in the vertical with $e^{\pm imz}$, where now $m^2 \equiv +R^2$. The particular solution

$$\hat{w} = B \sin[m(h+z)]$$

satisfies the equation and bottom boundary condition at $z = -h$. The top boundary condition then implies

$$\omega^2 = gk^2 \frac{\tan(mh)}{m}$$

Note:

$$m^2 = R^2 = \frac{k^2(N^2 - \omega^2)}{\omega^2}.$$

is just the standard dispersion relation for plane (unbounded) internal waves in a different format. However, in this case m cannot take arbitrary values, like in the modal solutions.

13.3.3 Unified treatment of dispersion relationsihp

Normally, we regard k as given and solve for $\omega(k)$. Now, because of the algebraic complexity, it is easier to regard ω as given and solve for $k = k(\omega)$.

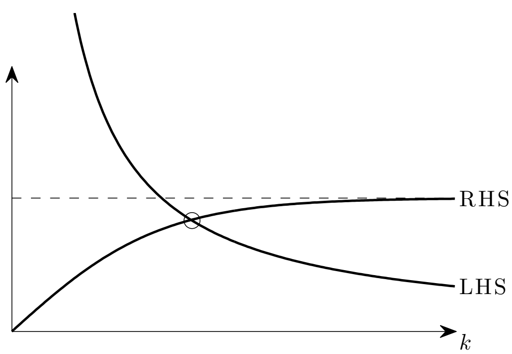
Thus, let ω be given. First assume that $\omega^2 > N^2$. Then $R^2 < 0$ and we must solve

$$\frac{\omega^2}{gk} = \frac{\tanh\left[\left(\frac{\omega^2 - N^2}{\omega^2}\right)^{1/2} kh\right]}{\left(\frac{\omega^2 - N^2}{\omega^2}\right)^{1/2}} \tag{13.16}$$

Regarding this as

$$\text{LHS} = \text{RHS}$$

we graph:



There is exactly one value of k for each $\omega^2 > N^2$.

If $\omega^2 \gg N^2$ we obtain $\omega^2/(gk) = \tanh kh$, which is the general dispersion relation for surface waves in homogeneous fluid ($N^2 = 0$). In the opposite limit, $\omega^2 \to (N^2)^+$, we obtain $\omega^2/(gk) = kh$, the dispersion relation for shallow water ($kh \ll 1$) surface waves. Now, however, this dispersion relation holds when $\omega^2 \to N^2$ *whether or not* $kh \ll 1$. The explanation is that as $\omega \to \pm N$, the vertical acceleration and buoyancy terms cancel. In the ocean, however, N^2 is so small that $\omega^2 = N^2 = ghk^2 \implies k^2h^2 \sim 10^{-3}$, i.e., at these frequencies the fluid would be in the shallow water limit anyway.

Dispersion relationship for $\omega^2 < N^2$

Now suppose that $\omega^2 < N^2$. In this case we will obtain multiple solutions for k, each corresponding to a different vertical wavenumber. With $R^2 > 0$ we must solve

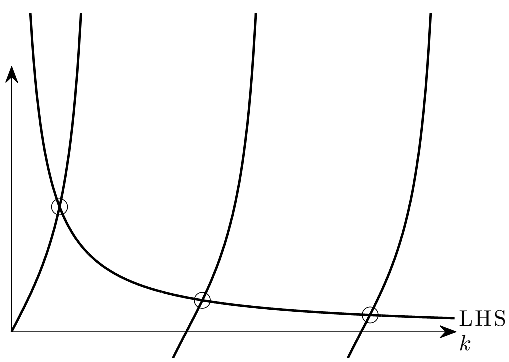
$$\frac{\omega^2}{gk^2} = \frac{\tan\left[mh\right]}{m}$$

or

$$\frac{\omega^2}{gk} = \frac{\tan\left[\sqrt{\frac{N^2-\omega^2}{\omega^2}}kh\right]}{\sqrt{\frac{N^2-\omega^2}{\omega^2}}}$$

again equate the left and right hand side,

$$\text{LHS} = \text{RHS}$$



First we consider small $mh \ll 1$, this yields

$$\frac{\omega^2}{gk^2} = \frac{mh}{m} = h, \quad \Rightarrow \omega^2 \approx (gh)k^2$$

Thus, the solution with the lowest value of m (or k) now corresponds to shallow water waves with phase speed $(gh)^{1/2}$. This is true provided that $\frac{N^2h}{g} \ll 1$. Thus,

> Barotropic shallow water surface gravity waves are essentially unaffected by the ocean's stratification. They are the same, at all wavenumbers, as they would be in a homogeneous fluid.

The other solutions in the case $\omega^2 < N^2$ correspond to internal waves with vertical modal structure. In the limit of very large g, the left hand side $\omega^2/(gk) \to 0$ (recall $\omega^2/(gk) = 1$

for deep water surface gravity waves) for and so the right-hand-side must also go to zero and the dispersion relationship becomes

$$\tan(mh) = \tan\left[\sqrt{\frac{N^2-\omega^2}{\omega^2}}kh\right] = 0$$

which implies that

$$\sqrt{\frac{N^2-\omega^2}{\omega^2}}kh \approx n\pi \quad , \quad n = 1, 2, 3, \ldots \tag{13.17}$$

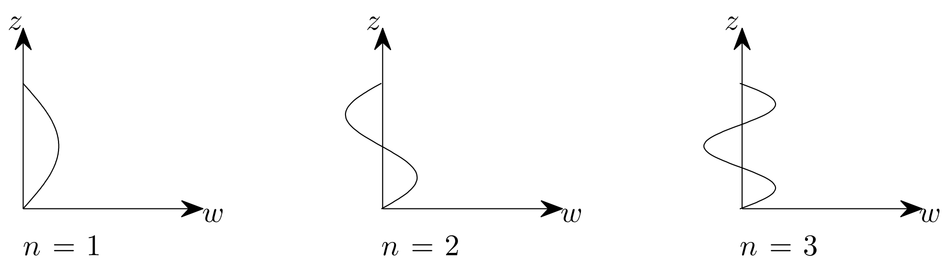
This can be rearranged in the form

$$(N^2-\omega^2)k^2 = m^2\omega^2$$
$$m^2 \equiv \frac{n^2\pi^2}{h^2}$$

which is the dispersion relation for internal waves with vertical wavenumber $m = n\pi/h$. The vertical velocity is

$$\hat{w} = \sin\left[\frac{n\pi}{h}(h+z)\right]$$

and has $n-1$ zero crossings as we saw in Chapter 11.



The approximation (13.17) corresponds to replacing the exact surface boundary condition by $w = 0$ (a rigid lid). The approximation is worst for the lowest ($n = 1$) mode. For that mode we have

$$\omega^2 \approx \frac{N^2k^2}{(k^2+\frac{\pi^2}{h^2})}.$$

This approximation is correct provided that

$$\frac{\omega^2}{gk}\sqrt{\frac{N^2-\omega^2}{\omega^2}} \ll 1$$
$$\Longrightarrow \frac{\omega^2}{gk}\left(\frac{\pi}{kh}\right) \ll 1$$
$$\Longrightarrow \omega^2 \ll ghk^2$$
$$\Longrightarrow \frac{N^2k^2}{(k^2+\frac{\pi^2}{h^2})} \ll ghk^2$$

which is obviously true for

$$\frac{N^2k^2}{\frac{\pi^2}{h^2}} \ll ghk^2$$

or restated another way

$$\boxed{\frac{N^2h}{g} \ll 1},$$

which is a constraint we've seen before in developing the Boussinesq approximation. In this limit, then, surface waves behave as if $N^2 = 0$, and internal waves behave as if there were a rigid lid.

13.3.4 Surface expression of internal waves

Internal waves do have a surface expression that can be inferred from satellite altimetry (Zhao papers). We can rewrite the dispersion relationship for modal internal waves with a free surface as,

$$\frac{\omega^2}{gk} = \frac{\tan(\alpha kh)}{\alpha} \tag{13.18}$$

where $\alpha = (N^2 - \omega^2)^{1/2}/\omega$. For small N^2h/g the zeros of (13.18) will be near $n\pi$ $(n = 0, 1, 2,)$. So to approximatly solve (13.18), we can use a Taylor's series expansion! The Taylor's series expansion of $\tan(x)$ near $n\pi$ is

$$\tan(x) = 0 + (x - n\pi) + \ldots$$

thus

$$\alpha kh = n\pi + \frac{\alpha\omega^2}{gk} \tag{13.19}$$

which can be re-arranged to a quadratic equation for k.

Thus, given h, N and ω and the mode number, one can calculate and approximate k and then $m = k(N^2 - \omega^2)^{1/2}/\omega$. The vertical velocity at the surface $(z = 0)$ is then

$$\hat{w} = \sin(mh) \tag{13.20}$$

and as $\partial\eta/\partial t = w$ one can then calculate the free-surface dispacement expected. This is in the homework.

13.4 Problem Set

1. For an infinetly deep ocean with $N = 10^{-2}$ rad/s and surface gravity waves of $T = 20$ s,

   (a) Calculate k and the horizontal wavelength

   (b) Calculate the vertical decay scale $\tilde{k}$.

2. Consider an ocean of depth $h = 1$ km with constant buoyancy frequency $N = 2\pi \times 10^{-3}$ rad/s on a non-rotating ocean

   (a) What is the horizontal wavelength of the mode-one semi-diurnal (*i.e.*, period of 12 h) internal tide? For both rigid surface at $z = 0$ and a non-rigid surface?

   (b) What is the horizontal wavelength of a mode-one internal wave with period of 1 h? For both rigid surface at $z = 0$ and a non-rigid surface?

   (c) For a semi-diurnal model-one internal wave in $h = 1000$ m depth, estimate the free surface displacement of the mode-one semi-diurnal internal tide with mid-water column maximum vertical isotherm dispacement of $\Delta z = 10$ m.

Chapter 14

Shallow water equations

In Chapter 11, we've seen that for the primitive equations with stratification $N^2(z)$, that if $kh \ll 1$ then the vertical momentum equation is essentially a balance between vertical pressure gradient and buoyancy (*i.e.,* $\partial p/\partial z = -g\rho'$) otherwise known as the *hydrostatic approximation*. From this we saw that all mode n internal waves were *non-dispersive*. We now start with the *hydrostatic* primitive equations and from that derive the shallow water equations, which will result in non-dispersive system throughout.

14.1 Linear and hydrostatic primitive equations

We assume a Boussinesq and hydrostatic fluid with density of $\rho_0(z)+\rho$ on an f-plane with no friction. The linear (small Rossby number for rotation) primitive equations (see Chapter 8) with a free surface at $z = \eta(x, y, t)$ and a bottom at $z = -h(x, y)$ are (x, y, z momentum, continuity, and density evolution),

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{14.1}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \tag{14.2}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{g}{\rho_0}\rho \tag{14.3}$$

$$u_x + v_y + w_z = 0 \tag{14.4}$$

$$\frac{\partial \rho}{\partial t} + w\frac{\partial \rho_0(z)}{\partial z} = 0 \tag{14.5}$$

where $N^2(z) = -(g/\rho_0)\frac{\partial \rho_0}{\partial z}$. Note that the density and vertical momentum equation can be combined so that

$$\rho_0^{-1}\frac{\partial^2 p}{\partial t \partial z} = -wN^2(z) \tag{14.6}$$

which we will see again later in this chapter.

14.2 Barotropic shallow water equations

Here we derive the barotropic shallow water equations where it is assumed that the density is a constant.

14.2.1 Momentum equations

For a fluid of constant density $\rho = \rho_0$, the hydrostatic approximation implies that $\partial p/\partial z = -g\rho_0$ which can be integrated from z to $z = \eta$ assuming that $p = 0$ at $z = \eta$ then,

$$p(z) = g\rho_0(\eta - z).$$

Because of this, the horizontal pressure gradient is not a function of the vertical,

$$\frac{\partial p}{\partial x} = g\rho_0 \frac{\partial \eta}{\partial x}, \quad \frac{\partial p}{\partial y} = g\rho_0 \frac{\partial \eta}{\partial y}.$$

Thus the horizontal momentum equations can be written as

$$\begin{aligned} \frac{\partial u}{\partial t} - fv &= -g\frac{\partial \eta}{\partial x} \\ \frac{\partial v}{\partial t} + fu &= -g\frac{\partial \eta}{\partial y} \end{aligned}$$

for $kh \ll 1$.

14.2.2 Continuity Equation

Next we derive the continuity equation for the shallow water equations taking the incompressible fluid continuity equation $\nabla \cdot \mathbf{u} = 0$ and vertically integrating,

$$\int_{-h}^{\eta} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \mathrm{d}z + w|_{z=\eta} - w|_{z=-h} = 0. \tag{14.7}$$

Assuming a flat bottom $w = 0$ at $z = -h$ and knowing the kinematic boundary conditions at the free surface ($z = \eta$) $D\eta/Dt = w$. Now we know from Leibniz's rule that

$$\int_{-h}^{\eta} \frac{\partial u}{\partial x} \mathrm{d}z = \frac{\partial}{\partial x} \left(\int_{-h}^{\eta} u \, \mathrm{d}z \right) - u\frac{\partial \eta}{\partial x}. \tag{14.8}$$

which implies that (in one dimension)

$$\begin{aligned} \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} - u\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x} \left[\int_{-h}^{\eta} u \, \mathrm{d}z \right] &= 0 \\ \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \left[(h + \eta) u \right] &= 0 \end{aligned}$$

where the last step is because the velocity is depth uniform. The form for the full shallow water equation continuity equation takes the vector form:

$$\frac{\partial \eta}{\partial t} + \nabla \cdot [(\eta + h)\mathbf{u}] = 0 \tag{14.9}$$

Where now $\mathbf{u}$ is the horizontal velocity vector. Note that this is the full (nonlinear) shallow water continuity equation. *Question: Was it necessary to assume that the depth is flat (i.e., h is constant)? If $h = h(x)$, what is the form of the bottom boundary condition?*

14.2.3 Linearizing the continuity equation

In the problem set from Chapter 2, you derived a nondimensional parameter that had to be small for linear shallow water waves. Here we re-examine that from another point of view of the continuity equation (14.9), which has a nonlinearity of $\eta\mathbf{u}$. Lets suppose $\eta = a\tilde{\eta}$ where $\tilde{\eta}$ is non-dimensional, and that h is flat and constant. Then we can rewrite the term

$$\nabla \cdot [(\eta + h)\mathbf{u}] = h\nabla \cdot [(1 + (a/h)\tilde{\eta})\mathbf{u}] \tag{14.10}$$

This implies that if $a/h \ll 1$, then we can neglect η part of this term and write the linear continuity equation,

$$\frac{\partial \eta}{\partial t} + \nabla \cdot [h\mathbf{u}] = 0 \tag{14.11}$$

This simplificaion is analogous to vertically integrating the wave kinetic energy only to $z = 0$ and not $z = \eta$. More formally, we are dropping terms of higher order a/h. *Question: What does this imply for the momentum equation which we assumed were linear apriori? How big is uu_x relative to u_t in the momentum equations?*

14.2.4 Complete Linear Shallow Water Equations

Thus the complete linear shallow water equations are

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{14.12a}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{14.12b}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0 \tag{14.12c}$$

The linear shallow water equations with rotation will be discussed in the next chapter. Recall that here that the fluid is homogenous (constant density ρ_0) and there is no *friction*.

14.2.5 Analogy to Incompressible Fluid?

It is worth considering if there are any analogs between the underlying acoustic (6.20–6.22) and shallow water equations (14.12a–14.12c). See the problem sets.

14.3 Constant depth and constant stratification: mode equations

This section follows that of Chapman and Rizzoli (Chapter 5). Assume now that (1) the depth is uniform, (2) the stratification is constant so N^2 is a constant, and (3) that the sea-surface is a rigid lid so that $\partial\eta/\partial t = w(z=\eta) = 0$. Note that only assumption (1) is necessary here. Assumptions (2) and (3) make the problem simpler.

Using wisdom gained from the modal internal wave solution, we choose a strategy of a separable solution of the form

$$
\begin{aligned}
u(x,y,z,t) &= U(x,y,t)F(z),\\
v(x,y,z,t) &= V(x,y,t)F(z),\\
w(x,y,z,t) &= W(x,y,t)G(z),\\
p(x,y,z,t) &= P(x,y,t)H(z),
\end{aligned}
$$

where F and G are non-dimensional, and U, V, and W have units of $\mathrm{m\,s^{-1}}$. Now with $w=0$ at $z=0$ and $z=-h$ this means that $G(z)$ has to satisfy $G=0$ at $z=0$ and $z=-h$. Plugging these solutions into (14.1–14.4) and (14.6), we get

$$\left(\frac{\partial U}{\partial t} - fV\right)F(z) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}H(z) \tag{14.13}$$

$$\left(\frac{\partial V}{\partial t} + fU\right)F(z) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}H(z) \tag{14.14}$$

$$\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)F(z) + W\frac{\mathrm{d}G}{\mathrm{d}z} = 0 \tag{14.15}$$

$$N^2 W G(z) = \frac{1}{\rho_0}\frac{\partial P}{\partial t}\frac{\mathrm{d}H}{\mathrm{d}z}, \tag{14.16}$$

where the last equation is based on (14.6) $\rho_o^{-1}p_{tz} = -wN^2$. Now this is starting to look like the modal internal wave solutions with $kh \ll 1$. Recall there that $w \propto \sin[(n\pi/h)(z+h)]$ and u and p had vertical structure of cos(). We see that if we assume here that u, v, and p have similarly shaped modal structure such that $H(z) = g\rho_0 F(z)$ then

$$
\begin{aligned}
\frac{\partial U}{\partial t} - fV &= -g\frac{\partial P}{\partial x}\\
\frac{\partial V}{\partial t} - fU &= -g\frac{\partial P}{\partial y}
\end{aligned}
$$

which looks a lot like the shallow water equations but this is for *internal* motions where P now takes the place of η, which implies that P has units of length. Now in order to satisfy continuity (14.15), we must have that $\mathrm{d}G/\mathrm{d}z = F/d$ where d is some scale depth, which we do not yet know.

Now if we think of P as the height of an isopycnal or isotherm then it would make sense that $W \propto \partial P/\partial t$ (the units match) and we can see from (14.16) that

$$N^2 W G(z) = -\frac{1}{\rho_0}\frac{\partial P}{\partial t}\frac{\mathrm{d}H}{\mathrm{d}z} = -g\frac{\mathrm{d}F}{\mathrm{d}z}\frac{\partial P}{\partial t}$$
$$= -gd\frac{\mathrm{d}^2 G}{\mathrm{d}z^2}\frac{\partial P}{\partial t}$$

so now by equating $W = \partial P/\partial t$ we get

$$\frac{\mathrm{d}^2 G}{\mathrm{d}z^2} + \frac{N^2}{gd}G = 0 \tag{14.17}$$

which must match the top and bottom boundary condtions that $G = 0$ at $z = 0, -h$. This becomes an eigenvalue problem where

$$G(z) = \sin\left(\frac{n\pi}{h}(z+h)\right)$$

where the eigenvalues d_n are defined as so that

$$\frac{N^2}{gd_n} = \frac{n^2\pi^2}{h^2}$$

or

$$d_n = \frac{N^2 h^2}{gn^2\pi^2} \tag{14.18}$$

such that (14.17) becomes

$$\frac{\mathrm{d}^2 G}{\mathrm{d}z^2} + \frac{n^2\pi^2}{h^2}G = 0.$$

We will come back to the interpretation of d_n (14.18) soon.

Lastly the continuity equation becomes,

$$\frac{\partial P}{\partial t} + d_n\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) = 0$$

where P replaces η and an effective depth d_n (14.18) replaces the water depth h.

14.3.1 Full Modal SWE and the Effective Depth

The combined modal shallow water equations that govern linear hydrostatic motions in the ocean are, for a particular mode number $n = 1, 2, ..,$

$$\frac{\partial U_n}{\partial t} - fV_n = -g\frac{\partial P_n}{\partial x} \tag{14.19a}$$

$$\frac{\partial V_n}{\partial t} + fU_n = -g\frac{\partial P_n}{\partial y} \tag{14.19b}$$

$$\frac{\partial P_n}{\partial t} + d_n\left(\frac{\partial U_n}{\partial x} + \frac{\partial V_n}{\partial y}\right) = 0 \tag{14.19c}$$

$$d_n = \frac{N^2 h^2}{g n^2 \pi^2} \tag{14.19d}$$

$$G_n(z) = \sin\left(\frac{n\pi}{h}(z+h)\right) \tag{14.19e}$$

$$F_n(z) = \cos\left(\frac{n\pi}{h}(z+h)\right). \tag{14.19f}$$

Recall that the zero mode ($n = 0$) - or barotropic mode - is not considered here because we assumed a rigid surface ($w = 0$ at $z = 0$).

Next we consider the meaning of d_n. Recall that for shallow water waves the phase speed $c = (gh)^{1/2}$ and for shallow-water modal internal waves $c_n = Nh/(n\pi)$. We see then by analogy that d_n is an "effective-depth" such that

$$c_n^2 = \left(\frac{Nh}{n\pi}\right)^2 = (gd_n)$$

Thus one interprtation of d_n is that it is the effective depth of a shallow-water wave that propagates with the shallow-water modal internal wave phase speed. Now expand upon assumptions that N^2 is constant and $w = 0$ at the surface. Both can be relaxed and similar solutions are derived.

14.4 Problem Set

1. For the barotropic SWE, was it necessary to assume that the depth is flat (*i.e., h* is constant in (14.9)? Show your work.

2. Consider that for a constant N^2 in a Boussinesq fluid, we have

$$N^2 = -(g/\rho_0)\frac{\Delta\rho}{h}.$$

   where $\Delta\rho \ll \rho_0$.

   (a) For $n = 1$, is d_n greater than or less than h?

   (b) Is it true that $d_m > d_n$ for $m < n$?

   (c) If the phase speed $c = (gd_n)^{1/2}$, can this be written instead as a "reduced gravity" phase speed $c = (g'h)^{1/2}$? What would the reduced gravity be in this case?

3. In order to assume hydrostatic motions, what constraints must be applied to the horizontal and vertical scales of motion?

4. Analogy between compressible fluid and SWE: Examine the compressible, no gravity, and no stratification, equations used for acoustics (6.20–6.21) and the shallow water equations (14.12a–14.12c) with **no rotation,** $f = 0$. In compressible fluid, $p = c^2\rho$, eliminating one equation. By making an analogy between p and η can you infer that the shallow water phase speed $c = (gh)^{1/2}$?

5. (EXTRA CREDIT) In 14.19, nonlinear terms were neglected. How would you represent the terms uu_x in (14.19) for mode number $n = 2$? Can nonlinearity force energy from one mode to another? If so, can you divine a simple algebra rule that says two modes numbered p and q can force mode n?

Chapter 15

Shallow water equations without rotation

15.1 Linear Shallow Water Equations without Rotation

Thus the complete linear shallow water equations without rotation (setting $f = 0$ in Eq. 14.12) are,

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{15.1a}$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y} \tag{15.1b}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0 \tag{15.1c}$$

The linear shallow water equations with rotation will be discussed in the next chapter. Recall that here that the fluid is homogenous (constant density ρ_0) and there is no *friction*.

15.2 Deriving a wave equation for shallow water

With the shallow water equations we can derive a shallow water wave equation. Taking the time derivative of (15.1c) and substituting in the momentum equation (15.1a and 15.1b) we get,

$$\eta_{tt} = \nabla \cdot [gh(x, y)\nabla\eta] \tag{15.2}$$

which is starting to look like the classic wave equation. If we assume (which we will not always do) that h is constant in space then we get

$$\eta_{tt} - gh\nabla^2\eta = 0 \tag{15.3}$$

which is the wave equation where, $c^2 = gh$ is the shallow water phase speed squared. Note, we previously derived this phase speed in Chapter 2 from potential flow theory and then assuming shallow water $kh \ll 1$.

15.3 Plane wave solutions and dispersion relationship

In the case where h is constant, we recover the dispersion relationship from the wave equation (15.3),

$$\omega^2 = gh(k^2 + l^2)$$

We can also plug a plane wave solution, *i.e.*, $\eta = \hat{\eta}\exp[\mathrm{i}(kx + ly - \omega t)]$ into (15.1a–15.1c) to get,

$$-\mathrm{i}\omega\hat{u} = -g\mathrm{i}k\hat{\eta}, \tag{15.4}$$

$$-\mathrm{i}\omega\hat{v} = -g\mathrm{i}l\hat{\eta}, \tag{15.5}$$

$$-\mathrm{i}\omega\hat{\eta} + h(\mathrm{i}k\hat{u} + \mathrm{i}l\hat{v}) = 0. \tag{15.6}$$

Multipy the 3rd line by $\mathrm{i}\omega$ and substitute in with the 1st two equations gives,

$$\omega^2\hat{\eta} - [gh(k^2 + l^2)]\hat{\eta} = 0,$$

which gives us the dispersion relationship again,

$$\omega^2 = gh(k^2 + l^2). \tag{15.7}$$

The point here is that we can use either the 2nd order wave equation OR the shallow water equations.

For a non-rotating plane wave propagating in the $+x$ direction, we also see that from (15.4),

$$\hat{u} = \left(\frac{g}{h}\right)^{1/2}\hat{\eta}, \tag{15.8}$$

which can be genralized for any arbitary plane wave direction.

15.4 Basic Diffraction

15.4.1 What is Diffraction?

[FROM WIKIPEDIA] Diffraction refers to various phenomena which occur when a wave encounters an obstacle or a slit. It is defined as the bending of light around the corners of an obstacle or aperture into the region of geometrical shadow of the obstacle. Diffraction

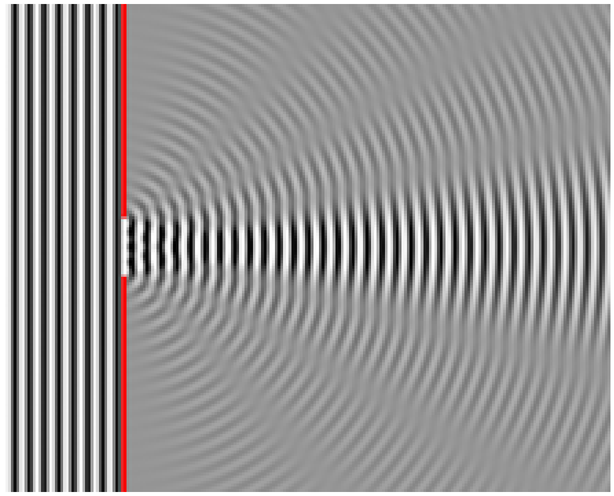


Figure 15.1: Example of diffraction through a slit.

occurs with all waves, including sound waves, water waves, and electromagnetic waves such as visible light, X-rays and radio waves.

While diffraction occurs whenever propagating waves encounter such changes, its effects are generally most pronounced for waves whose wavelength is roughly comparable to the dimensions of the diffracting object or slit. If the obstructing object provides multiple, closely spaced openings, a complex pattern of varying intensity can result. This is due to the addition, or interference, of different parts of a wave that travels to the observer by different paths, where different path lengths result in different phases (see diffraction grating and wave superposition). The formalism of diffraction can also describe the way in which waves of finite extent propagate in free space.

We're not formally covering diffraction here, but the basic idea of waves propagating through a very small slit can be thought of as solutions to the wave equation in polar coordinates. We will explore that here.

15.4.2 Setup

Now consider a very thin piston at $(x, y) = (0, 0)$ pumping the surface up and down at a frequency ω. How does the surface respond? We start with the wave equation and transform it into polar coordinates where $(x, y) \to (r, \theta)$ where $x = r\cos\theta$ and $y = r\sin\theta$. We assume that the solution to the wave equation (15.3) is uniform in the θ direction and so

$$\nabla^2 f = \partial_r^2 f + (1/r)\partial_r f$$

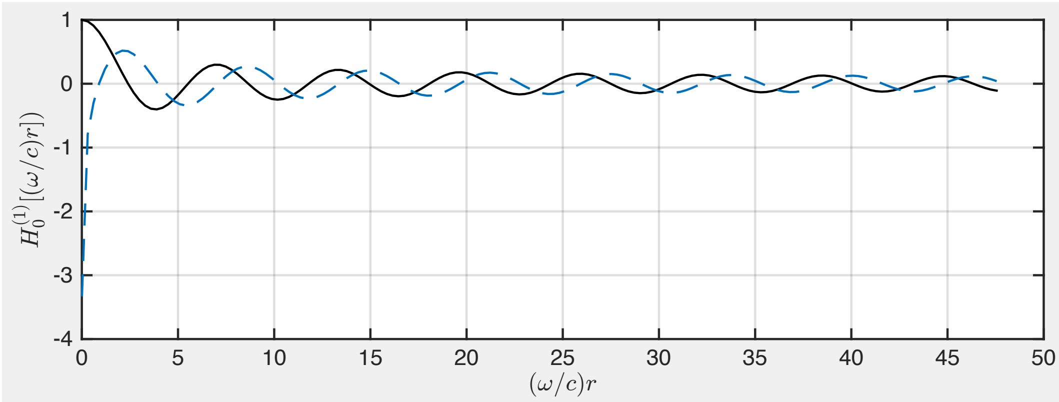


Figure 15.2: Hankel function $H_0^{(1)}[(\omega/c)r]$ as a function of $(\omega/c)r$ for both real (solid) and imaginary (dashed) parts. Note how the r decay is rapid at first and then slows down to a slowly-decaying cosine function.

and that the solution goes like $e^{\mathrm{i}\omega t}$ results in

$$\begin{aligned} -\omega^2\hat{\eta} &= gh\nabla^2\hat{\eta} \\ &= (gh)\left[\frac{\partial^2\hat{\eta}}{\partial r^2} + \frac{1}{r}\frac{\partial\hat{\eta}}{\partial r}\right] \end{aligned}$$

Which can be reworked into a form as (with $c^2 = gh$),

$$r^2\hat{\eta}_{rr} + r\hat{\eta}_r + \frac{\omega^2}{c^2}r^2\hat{\eta} = 0. \tag{15.9}$$

Note that (15.9) is a 2nd order homogenous ordinary differential equation but not with constant coefficients. In general, solving such equations is a pain. However, (15.9) is in the form of a Sturm-Liouville eigenfunction problem. Recall that ordinary differential equations of the form

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial\phi}{\partial x}\right) + q(z)\lambda\phi = 0$$

is a special case of a Sturm-Liouville eigenfunction problem.

Boundary Conditions

What about boundary conditions? We need two boundary conditions for $\hat{\eta}$. The first obvious boundary condition is that the solution $\hat{\eta}$ must decay at large r or $\hat{\eta} \to 0$ as $r \to \infty$.

What about at $r = 0$, where the piston goes up or down? We can approximate the piston as having a tiny but finite width near $(x, y) = (0, 0)$ (or near $r = 0$). At this location the piston oscillates harmonically and so near $r = 0$, $\eta(t)\hat{\eta}(r)e^{\mathrm{i}\omega t}$? Now, we could do the problem with a bounary condition at some small positive r_+ but the issue is that we would have to match the boundary condition to a linear combination of the two solutions to the fundamental ODE. Instead we will bring the boundary condition direction to $r = 0$ and assign it as $\hat{\eta}(r) = \delta(r)$ so that

$$\eta(r = 0, t) = \delta(r) \cos(\omega t).$$

With (15.9) and the boundary conditions, this makes a full statement of the Sturm-Liouville eigenvalue problem and the solution has a number of properties.

15.4.3 Bessel function as a solution

The wave equation in polar coordinates (15.9) with the boundary condition looks a lot like the ordinary differential equation for Bessel functions

$$x^2 y'' + xy' + (x^2 - n^2)y = 0. \tag{15.10}$$

with solutions as Bessel functions of the first and second kind $J_n(x)$, $Y_n(x)$. Now for our equation (15.9), $n = 0$ and so $J_0(x)$ and $Y_0(x)$ will be our solutions. Note that at $x = 0$, J_0 is finite and Y_0 is singular. These standard Bessel functions are used for standing type solutions, ie $J_0(x)\cos(\omega t)$. In addition, the standard Bessel functions can be linearly combined into something called a Hankel function $H_0^{(1)}$ and $H_0^{(2)}$ that correspond to $J_0(x)$ and $Y_0(x)$ and are defined as $H_0^{(1)}(x) = J_0(x) + \mathrm{i}Y_0(x)$ and $H_0^{(1)}(x) = J_0(x) - \mathrm{i}Y_0(x)$. One can think of Hankel functions as complex exponentials and they are used to get propagating solutions. *i.e.*,

$$H_0^{(1)}(x) \exp(\mathrm{i}\omega t) \tag{15.11}$$

15.4.4 Conversion to Bessel Function Equation

To convert (15.9) to the ODE for the Bessel function we have to create a new variable $s = (\omega/c)r$ and $\mathrm{d}s = (\omega/c)\,\mathrm{d}r$ so that $r = (c/\omega)s$ and $\mathrm{d}r = (c/\omega)\,\mathrm{d}s$. With this substitution (15.9) becomes

$$s^2 \hat{\eta}_{ss} + s\hat{\eta}_s + s^2 \hat{\eta} = 0.$$

which matches (15.10) with $n = 0$ and the solution is $J_0(s)$.

15.4.5 Solution

Thus, the solution in r space is $\hat{\eta}(r) = H_0^{(1)}[(\omega/c)r]$. Now it is worth noting that for plane waves, by definition, $\omega/c = k$ and so this could be interpreted as $H_0^{(1)}(kr)$. However, a wavenumber k is technically not well defined for things that are not plane waves, as here we are in cylindrical or polar coordinates The solution for $\hat{\eta}(r)$ as a function of $(\omega/c)r$ is shown

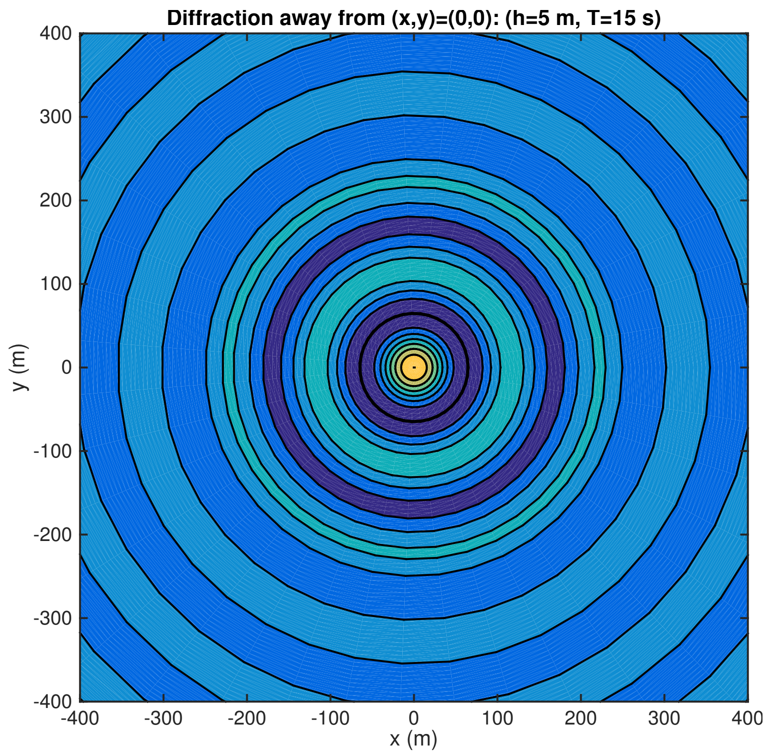


Figure 15.3: Snapshot of the sea surface for the piston forcing at $(x, y) = (0, 0)$ m with water depth $h = 5$ m and $T = 15$ s.

in Figure 15.2. Note that for $(\omega/c)r > 15$, the solution looks like a slowly decaying cosine function. In fact for large arguments $H_0^{(1)} \approx (2/(\pi x))^{1/2} \cos(x - \pi/4)$ which is consistent with Figure 15.2.

The full solution for $\eta(r, t)$ is,

$$\eta(r,t) = \mathrm{Re}\left\{ H_0^{(1)}[(\omega/c)r]e^{\mathrm{i}\omega t} \right\},$$

a snapshot of which is shown in in Figure 15.3.

15.5 Energy Conservation

Here, we will demonstrate energy conservation for the linear shallow water equations with a flat bottom (h is constant). Taking the LSWE (15.1),

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x},$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y},$$

$$\frac{\partial \eta}{\partial t} + h\left[\frac{\partial}{\partial x}(u) + \frac{\partial}{\partial y}(v)\right] = 0,$$

and multiplying these 3 equations by $(hu, hv, g\eta)$, respectively, integrating by parts and summing, results in,

$$\frac{h}{2}\frac{\partial}{\partial t}\left[u^2 + v^2 + g\eta^2\right] = -gh\left[\left(\eta\frac{\partial u}{\partial x} + u\frac{\partial \eta}{\partial x}\right) + \left(\eta\frac{\partial v}{\partial y} + v\frac{\partial \eta}{\partial y}\right)\right], \tag{15.12}$$

$$= -gh\frac{\partial(\eta u)}{\partial x} + \frac{\partial(\eta v)}{\partial y}. \tag{15.13}$$

We can rewrite this in vector form as,

$$\frac{\partial}{\partial t}\underbrace{\left[\frac{h}{2}|\mathbf{u}|^2 + \frac{1}{2}g\eta^2\right]}_{=E} + \nabla\cdot\underbrace{(gh\eta\mathbf{u})}_{=\mathbf{F}} = 0, \tag{15.14}$$

which is the instantaneous (unaveraged) energy conservation in flux conservation form,

$$\frac{\partial E}{\partial t} + \nabla\cdot\mathbf{F} = 0.$$

We've seen this kind of energy conservation now for arbitrary depth surface gravity waves, sound waves, and internal waves. It remains to be shown that for plane waves that $\langle\mathbf{F}\rangle = \langle E\rangle\mathbf{c}_g$. *Question: Does this instantaneous (unaveraged) form of energy conservation change with rotation?*

15.6 Reflection off of a boundary

Consider a plane shallow water wave of radian frequency ω in constant depth h propagating at angle θ_I to the $+x$ direction that impinges on a solid boundary at $x = 0$ that extends to $-\infty < y < \infty$. In acoustics and internal waves off of a flat bottom, the reflection was *specular* in that $\theta_R = -\theta_I$. Do shallow water plane waves also reflect specularly?

First we need to consider the appropriate boundary condition. It is no-flow though the boundary, $\mathbf{u}\cdot\mathbf{n} = 0$ where $\mathbf{n}$ is the outward normal from the boundary. In the case of a boundary at $x = 0$, it implies $u = 0$. From the x momentum equation, we have

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}.$$

Thus, if $u = 0$ then $\eta_x = 0$ at the boundary as well. For an arbitrary solid boundary, this can be generalized to $\nabla\eta\cdot\mathbf{n} = 0$ at the solid boundary. *Question: Does this also work on an f-plane?*

From this, we can now use the same machinery we used for acoustic waves. We write the incident wave η_I and reflected wave η_R as,

$$\eta_I(x, y, t) = \hat{\eta}_I \exp[\mathrm{i}(k_I x + l_I y - \omega_I t)],$$
$$\eta_R(x, y, t) = \hat{\eta}_R \exp[\mathrm{i}(k_R x + l_R y - \omega_R t)].$$

Now the boundary condition at $x = 0$ becomes,

$$\frac{\partial \eta_I}{\partial x} + \frac{\partial \eta_R}{\partial x} = \mathrm{i}k_I\hat{\eta}_I \exp[\mathrm{i}(l_I y - \omega_I t)] + \mathrm{i}k_R\hat{\eta}_R \exp[\mathrm{i}(l_R y - \omega_R t)] = 0. \tag{15.15}$$

In order for (15.15) to be true, a few things are necessary. First, $(l_I y - \omega_I t) = (l_R y - \omega_R t)$ for all time and all y and all t. In order for this to be true then $\omega_R = \omega_I$ and $l_R = l_I$. Then, since both incident and reflected wave obey the dispersion relationship (15.7) then $|k_I| = |k_R|$ which really implies that $k_R = -k_I$, otherwise the reflected wave would be propagating into the boundary still. This implies that $\theta_R = -\theta_I$. Lastly, then we see that $\hat{\eta}_R = \hat{\eta}_I$ and the wave is fully reflected. So thus, we have specular reflection again.

15.7 Propagation over a step

Now consider the case of a $+x$ direction propagating wave in depth h_1 that encounters at $x = 0$ a step in the bathymetry to h_2. This is analogous to the problem of sound wave reflection and transmission in the problem set of Chapter 6. Now, one requirement of the shallow water equations is that the bathymetry varies gently. A step clearly violates this as ∇h becomes very big. The way that this is dealt with is to assume that except for right near the step, the shallow water equations are fine. Then we need to have continuity relationships on each side of the step. This means that we need is to develop the boundary conditions to the problem using the shallow water equations.

First, we get a boundary condition for η by considering the x momentum equation $u_t = -g\eta_x$. Integrate this equation across the depth discontinuity at $x = 0$ to get,

$$\int_{-\Delta}^{\Delta} \frac{\partial u}{\partial t}\, \mathrm{d}zx = -g\left[\eta|_{\Delta} - \eta|_{-\Delta}\right], \tag{15.16}$$

$$2\Delta\frac{\partial u}{\partial t} = -g\left[\eta|_{\Delta} - \eta|_{-\Delta}\right]. \tag{15.17}$$

Thus, for $\partial u/\partial t$ to be finite as $\Delta \to 0$, we must have that $\eta(x = \Delta) = \eta(x = -\Delta)$ or that η is continuous across the boundary.

The boundary condition for velocity is interesting and is derived from the continuity equation $\eta_t = (hu)_x$. Again, integrating across the discontinuity gives,

$$\int_{-\Delta}^{\Delta} \frac{\partial \eta}{\partial t}\, \mathrm{d}zx = \left[(hu)|_{\Delta} - (hu)|_{-\Delta}\right], \tag{15.18}$$

$$2\Delta\frac{\partial \eta}{\partial t} = -g\left[(hu)|_{\Delta} - (hu)|_{-\Delta}\right]. \tag{15.19}$$

Thus, again for η_t to be finite as $\Delta \to 0$, we see that (hu) must be continuous or that $h_1 u_1 = h_2 u_2$, where u_1 and u_2 represent the velocity at either side of the bathymetry step.

With these two boundary conditions to be applied at a step, one can now do the reflection and transmission problem across a boundary. *Question: With the boundary condition* $h_1u_1 = h_2u_2$*, what quantity is* continuous *across the boundary?* In general this boundary condition can be written as $h\mathbf{u}\cdot\mathbf{n}$ is continous across the step where $\mathbf{n}$ is the outward normal from the step.

15.8 Closed Body of Water: An Eigenvalue Problem

15.8.1 Closed Boundary Conditions

For the linear shallow water, there are two types of boundary conditions, (i) closed boundary conditions and (ii) open boundary conditions. The former is relatively straightforward to handle the latter is subtle. For closed boundary conditions, the standard boundary condition is no normal flow into the boundary or,

$$\mathbf{u}\cdot\hat{\mathbf{n}} = 0; \tag{15.20}$$

as then,

$$\partial_t \mathbf{u}\cdot\hat{\mathbf{n}} = 0,$$

which implies that,

$$\nabla\eta\cdot\hat{\mathbf{n}} = 0. \tag{15.21}$$

If we use the 2D wave equation form (15.3) in a closed domain such as a lake. then this is the appropriate boundary condition. However, it turns (15.3) into an eigenvalue problem.

15.8.2 Setup of Problem

Consider (15.3),

$$\eta_{tt} = gh\nabla^2\eta,$$

in a domain G where on ∂G the boundary condition of $\nabla\eta\cdot\hat{\mathbf{n}} = 0$ is applied. Now we cannot use the plane wave (or the polar coordinate solutions). Instead we seek solutions of the form,

$$\eta(x,y,t) = \hat{\eta}(x,y)e^{-\mathrm{i}\omega t},$$

which transforms (15.3) to the Helmholtz equation,

$$\nabla^2\hat{\eta} + \frac{\omega^2}{gh}\hat{\eta} = 0, \tag{15.22}$$

with boundary condition $\nabla\hat{\eta}\cdot\hat{\mathbf{n}} = 0$. This is similar to the solutions for the vibrating string (wave equation) on a finite domain. There are only certain solutions that satisfy an eigenvalue problem - in particular for the boundary conditions only cos() and not sin() will satisfy the boundary conditions.

In general, there is only a solution to (15.22) for a discrete set of eigenvalues ω_n and eigenfunction $\hat{\eta}_n(x, y)$. For example, in a 1D case in a domain between $0 \leq x \leq L$ the frequencies are,

$$\omega_n = n\pi(gh)^{1/2}/L, \quad n = 1, 2, \ldots,$$

and,

$$\hat{\eta}(x)_n = \cos\left(\frac{n\pi x}{L}\right).$$

Consider another example: A circular domain. Then the eigenfunctions are (likely) Bessel functions in r and sines/cosines in θ. *Question: What happens if you force a closed body of water at a frequency corresponding to one of the eigenvalues ω_n?*

15.8.3 Aside on open boundaries

What about open boundaries. We have two options, either specify η or $\mathbf{u}$. This works fine for incoming waves. What happens with outgoing waves?

15.9 Friction and Damping

Up to now, we've only explicitly examined inviscid systems for surface gravity waves, acoustic waves, and internal waves. Here, we now examine frictional effects in the shallow water equations. Simple frictional effects are often represented in the shallow water momentum equations as Rayleigh friction, that is $-r(u, v)$ where r is the Rayleigh drag coefficient that has units of 1/Time. [*ASIDE: Is there real physics in this form of friction?*]

With Rayleigh friction, the resulting SWE are

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} - ru$$
$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y} - rv$$
$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0$$

We can re-arrange this into a new equation for η only. Taking the time derivative of the continuity equation and substituting from the momentum equations result in,

$$\eta_{tt} + \nabla \cdot (-gh\nabla\eta - rh\mathbf{u}) = 0$$

and then by subsituting in for $r\eta_t = -\nabla \cdot (rh\mathbf{u})$ we get

$$\eta_{tt} + \nabla \cdot (-gh(x, y)\nabla\eta) + r\eta_t = 0 \tag{15.23}$$

For constant h, we can now look for wave-like solutions, $\eta = \hat{\eta}(t)e^{ikx}$ so that,

$$\hat{\eta}_{tt} + r\hat{\eta}_t + \omega_0^2\hat{\eta} = 0, \tag{15.24}$$

where $\omega_0^2 = (gh)k^2$. now plugging in $\hat{\eta} = \exp(st)$ we get

$$s^2 + rs + \omega_0^2 = 0, \tag{15.25}$$

which has solutions,

$$s = \frac{-r \pm \sqrt{r^2 - 4\omega_0^2}}{2}. \tag{15.26}$$

Now if the dissipation rate is small, $r^2 \ll 4\omega_0^2$, then we can expand this using the fact that $(1+\epsilon)^{1/2} \approx 1 + \epsilon/2$,

$$s = -\frac{r}{2} \pm \mathrm{i}\omega_0\sqrt{1 - r^2/(4\omega_0^2)} \approx -\frac{r}{2} \pm \mathrm{i}\omega_0\left(1 - \frac{r^2}{8\omega_0^2}\right), \tag{15.27}$$

Thus the plane wave solution has propagating wave solution that decay slowly, *i.e.,*

$$\eta = \eta_0 \exp(-rt/2)\exp(\mathrm{i}(kx - \omega t),$$

with frequency

$$\omega = \omega_0\left(1 - \frac{r^2}{8\omega_0^2}\right),$$

that is slightly "de-tuned" from the normal dispersion relationship.

15.10 Problem Set

1. In the linear shallow water equations with a wave propagating in the $+x$ direction in constant depth h

   (a) What is the relationship between wave amplitude a and the cross-shore velocity magnitude u_0?

   (b) Is this relationship between a and u_0 the same as for linear surface gravity waves from Chapter 2?

2. Can you give a quick explanation why is the inertial-gravity wave velocity in the direction of the wavenumber $\mathbf{k}$ when for internal waves $\mathbf{u} \cdot \mathbf{k} = 0$?

3. For linear shallow water waves we neglected the uu_x term in

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x}$$

   Using the linear SWE solutions what is the size of the uu_x term relative to the other two terms?

4. Does the instantaneous (unaveraged) form of energy conservation (15.14) change with rotation added? Why or why not? Is this question familiar?

5. Shallow water wave reflection and transmission across a step bathymetry. Consider a wave of frequency ω propagating in depth h_1 at angle θ_I to a bathymetry step at $x = 0$ for $-\infty < y < \infty$ that has depth $h_2 > h_1$ on the other side of the step. The boundary conditions at the step are that η and hu are continuous. Consider the incident wave to have amplitude of one, $\hat{\eta}_I = 1$.

   (a) With the boundary condition $h_1 u_1 = h_2 u_2$, what is the quantity that is *continuous* across the boundary?

   (b) What are the reflected and transmitted wave amplitudes ($\hat{\eta}_R$ and $\hat{\eta}_T$) and angles (θ_R and θ_T)?

   (c) Can the relationship between θ_T and θ_I be generalized? What is it called?

   (d) for $h_2 > h_1$ can one have total reflection and if so what is the incident critical angle?

   (e) for $h_1 > h_2$ can one have total reflection and if so what is the incident critical angle?

6. For the boundary condition (15.20) and (15.21), show that the spatially-integrated energy is conserved when integrated over a closed body of water such as a lake.

7. Consider a rectangular lake of 50 m depth in a non-rotating world: How long must one dimension be in order to create resonant motions at

   (a) Semi-diurnal periods (imagine tidal forcing)

   (b) Diurnal periods (*e.g.,* sea-breeze)

   (c) Do any lakes or enclosed bodies of water exist that are this big?

   (HINT: use solutions from Problem set of Chapter 1.)

8. Rederive energy conservation for the linear shallow water equations to show that the addition of Rayleigh friction can only extract energy and not add energy.

Chapter 16

Shallow water equations with Rotation

Much of this Chapter is based on Hendershott (1980) and the Chapman and Malanotte-Rizolli notes in honor of Myrl.

16.1 Laplace's Tidal Equations (LTE)

To explain the dynamics of tides, Laplace in 1775-1776 derived equations for the evolution of a homogeneous thin fluid on a sphere subject to gravity. This is a remarkable achievement particularly for the time. Newton's *Principia* was only published less than 100 years before setting up the theory of mechanics and gravitation! The assumptions that go into the LTE are

1. linear equations that describe small perturbations on a state of rest
2. a perfect sphere
3. a uniform gravitational field
4. a rigid (non-deformable) crust
5. only the component of Coriolis normal to the geoid is considered.

The Hendershott (1980) review discusses how good these approximations are in detail with some surprising answers. With these assumptions, the LTE results in three equations for η, which are the free surface deviation from the mean, u_ϕ the zonal flow (along a constant latitude defined as θ) and meridional flow u_θ (along constant longitude defined as ϕ). Laplace's

Tidal equations are

$$\frac{\partial \eta}{\partial t} + \frac{1}{r\cos\theta}\left[\frac{\partial (hu_\phi)}{\partial \phi} + \frac{\partial (hu_\theta \cos\theta)}{\partial \theta}\right] = 0 \tag{16.1a}$$

$$\frac{\partial u_\phi}{\partial t} - 2\Omega \sin\theta u_\theta = -\frac{1}{r\cos\theta}\frac{\partial(\eta - \Gamma)}{\partial \phi} \tag{16.1b}$$

$$\frac{\partial u_\theta}{\partial t} + 2\Omega \sin\theta u_\phi = -\frac{1}{r}\frac{\partial(\eta - \Gamma)}{\partial \theta} \tag{16.1c}$$

Where h is the still water depth, r is the radius of the sphere, Ω is the rotation rate, and $\Gamma = \Gamma(\theta, \phi, t)$ is the tidal generating force - essentially the gravitational force of the moon and sun on the fluid of the sphere. These equations look familiar and bear a resemblance to the shallow water equations in a standard x-y coordinate system with analogies to the continuity equation and momentum equations.

The LTE accept a number of types of freely propagating "wave" solutions in one set of equations. These include

- Sverdrup/Poincare/Inertial-Gravity Waves
- Kelvin Waves
- Short and Long Rossby waves
- Equatorial Waves
- Bathymetrically trapped waves

The barotropic tide is a forced motions that can respond as a mixture of Kelvin and Inertial-Gravity waves.

16.2 Shallow water equations with Rotation on f-plane

Lord Kelvin in 1879 projected these equations on a plane of constant rotation rate to come up with the rotating shallow water equations. When these have a constant rotation rate, this is called the f-plane, and there are no rotation variations with latitude. Thus an f-plane can only cover a limited region of the ocean. Yet it is still super useful and allows for much insight to be derived.

The linear shallow water equations with rotation are similar to those without rotation only that we add in a $f\hat{\mathbf{k}} \times \mathbf{u}$ in the momentum equation to result in

$$\frac{\partial \mathbf{u}}{\partial t} + f\hat{\mathbf{k}} \times \mathbf{u} = -g\nabla\eta \tag{16.2}$$

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0 \tag{16.3}$$

or

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{16.4a}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{16.4b}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0 \tag{16.4c}$$

16.2.1 Boundary Conditions

At solid boundaries we need to have no normal flow or $\mathbf{u} \cdot \mathbf{n} = 0$. When we had no-rotation $f = 0$, the no-normal flow boundary conditions translated into $\nabla\eta \cdot \mathbf{n} = 0$ very cleanly. How do we reproduce this boundary conditions for a f-plane? We first take the ∂_t of (16.4a) to get

$$u_{tt} - fv_t = -g\eta_{xt}$$

and then substitute back (16.4b) to get

$$u_{tt} + f^2 u = -g \underbrace{(f\eta_y + \eta_{xt})}_{=0} \tag{16.5}$$

in order to ensure that $u = 0$ at solid boundary at a fixed x in (16.5) the quantity in underbrace must be zero. For a boundary at a constant y, a similar equation can be derived. This equation (16.5) is important because with rotation, it is the way to relate one velocity component to η. Without rotation (set $f = 0$ in Eq. 16.5) it clearly simpler.

This boundary condition can be generalized for arbitrary geometry as

$$\nabla\eta_t \cdot \hat{\mathbf{n}}_\perp + f\nabla\eta \cdot \hat{\mathbf{n}}_\parallel = 0, \tag{16.6}$$

and interpreted as the non-rotating boundary condition augmented by rotation. Consider, purely steady geostrophic flow at a boundary. Then in order for $u = 0$ then η_y must also be zero, as η_x only relates to the alongcoast (v) current. But when f is small, then the standard non-rotating boundary condition applies.

16.3 Potential vorticity on an f-plane of constant depth

Vorticity $\zeta = \nabla \times \mathbf{u} = v_x - u_y$, and we can derive a vorticity equation for a constant depth fluid by taking the curl of (16.2) which results in

$$\frac{\partial \zeta}{\partial t} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

where the first term is rate of change of relative vorticity and the 2nd term is the stretching of planetary vorticity. Substituting back the continuity equation yields

$$\frac{\partial}{\partial t}\left(\zeta - \frac{f\eta}{h}\right) = 0 \tag{16.7}$$

which is a linear barotropic constant depth potential vorticity conservation statement. This is an approximation to the more general potential vorticity conservations statement for the full nonlinear shallow water equations of

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{h+\eta}\right). \tag{16.8}$$

Question: Can you show how the linear potential vorticity conservation (16.7) comes from the full potential vorticity conservation?

16.4 Deriving a wave equation with rotation for shallow water

The linear PV conservation result will be used to now derive a wave equation for shallow water on an f-plane for flat bottom depth. The procedure is identical to without rotation. We start off taking the time-derivative of (16.3) and substituting into that (16.2) to get

$$\eta_{tt} = gh\nabla^2\eta + fh\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{16.9}$$

This equation almost has only η as a dependent variable with the exception of the presence of vorticity above. The vorticity can be turned into η using the linear potential vorticity (16.7) to get

$$\eta_{tt} + f^2\eta - gh\nabla^2\eta = 0 \tag{16.10}$$

which is a modified wave equation!

Note that this wave equation (16.10) is only valid for flat bottom and an f-plane. A more general wave equation that is valid for arbitrary bathymetry is derived similarly but one needs to take another time-derivative (TRY IT!). The result is

$$\eta_{ttt} + f^2\eta_t - \nabla\cdot[gh(x,y)\nabla\eta_t] + fg[(h\eta_x)_y - (h\eta_y)_x] = 0 \tag{16.11}$$

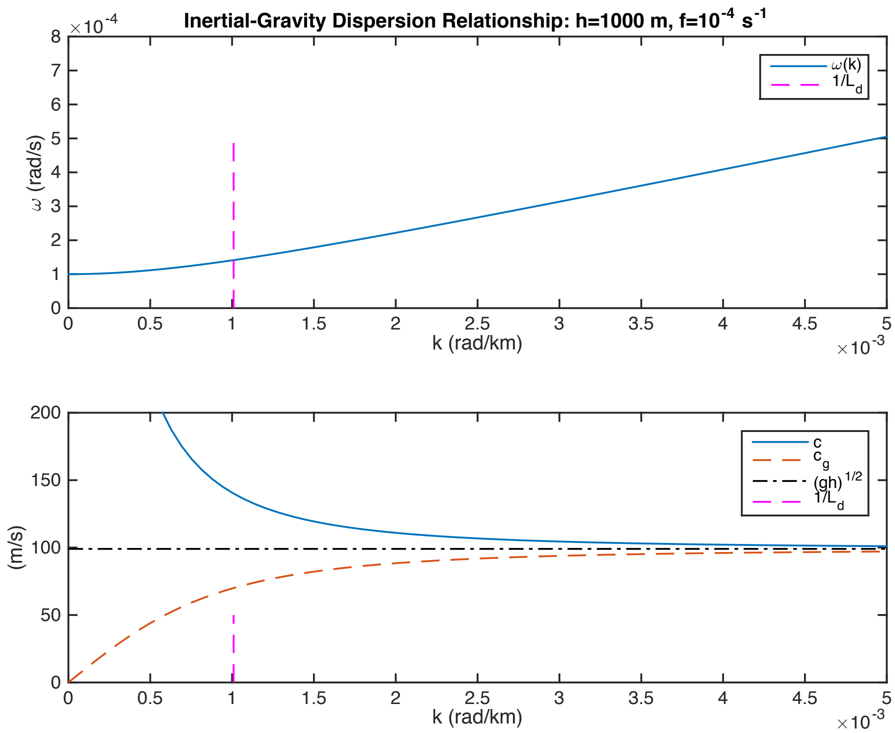


Figure 16.1: Inertial Gravity wave dispersion relationship (a) $\omega = [(gh)k^2 + f^2]^{1/2}$ for $h =$ 1000 m and $f = 10^{-4}$ s^{-1}, and (b) $c = \omega/k$ and $c_g = \partial\omega/\partial k$ versus k. The magenta dashed line indicates $(2\pi)L_d^{-1}$. The black dash-dot line in (b) indicates the non-rotating phase and group velocity $(gh)^{1/2}$.

16.5 Inertia-Gravity Waves: Plane wave solutions and dispersion relationship

In the case where h is constant, if we plug in a plane wave solution of $\eta = \eta_0 \exp[\mathrm{i}(kx - \omega t)]$ to (16.10), we recover the dispersion relationship:

$$\omega^2 = gh(k^2 + l^2) + f^2,$$

which is similar to without rotation but with an addition term of f^2. This is also clearly related to the modal internal wave dispersion relationship we have already derived. These waves are also called Sverdrup waves and in particular situations Poincare waves.

16.5.1 Phase speed and group velocity

The dispersion relationship describes a flat to linear shape for a wave in the $+x$ direction,

$$\omega = \sqrt{(gh)k^2 + f^2},$$

which is shown in Figure 16.1a. The group velocity is

$$c_g = \frac{\partial \omega}{\partial k} = \frac{(gh)k}{[(gh)k^2 + f^2]^{1/2}}$$

which is near-zero for very long wavelengths or length-scales (*i.e.,* $k \ll L_d^{-1}$). For short wavelengths (or short length-scales, $k \gg L_d^{-1}$), the dispersion relationship and phase speed approach the non-rotating limit of being non-dispersive (Figure 16.1b).

16.5.2 Solution for velocity

We can also then get a solution by plugging in a plane wave for η, u, and v into the f-plane shallow water equations, yielding

$$\begin{aligned} -\mathrm{i}\omega u_0 - f v_0 &= -\mathrm{i} g k \eta_0 \\ -\mathrm{i}\omega v_0 + f u_0 &= 0 \end{aligned}$$

Thus we see that $v_0 = -\mathrm{i}(f/\omega)u_0$ and then

$$\begin{aligned} -\mathrm{i}\left(\omega + \frac{f^2}{\omega}\right) u_0 &= -\mathrm{i} g k \eta_0 \\ u_0 &= \frac{gk\omega}{\omega^2 + f^2}\eta_0 \\ u_0 &= \frac{gk\omega}{ghk^2}\eta_0 \\ u_0 &= \frac{\omega}{hk}\eta_0 \\ u_0 &= \frac{c}{h}\eta_0 \end{aligned}$$

So we see here again that $u_0/c = \eta_0/h$, which again provides a simple physical constraint that $\eta_0/h < 1$. But we also already saw that for the linear shallow water equations. Then we also get the relationship that

$$v_0 = \frac{-\mathrm{i}f}{\omega} u_0$$

which means that v is $\pi/2$ out of phase with u and that $|v_0| \le |u_0|$. This also means that the particle motions with such a wave are ellipses with the major axis in the direction of the wavenumber $\mathbf{k}$.

16.5.3 Wave-averaged Energy Conservation

One can take the instantaneous energy conservation equation and wave-average it and using the solutions for velocity and the dispersion relationship derive a wave energy conservation equation

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(c_g E) = 0.$$

See problem set.

16.5.4 Reflection of Inertial Gravity Waves

Many wave systems have specular reflection off of a boundary? What about Inertial-gravity waves? Lets find out. Consider a boundary at $x = 0$ with an incident wave with $\mathbf{k} = (k, l)$. The boundary condition is (16.6)

$$\nabla\eta_t \cdot \hat{\mathbf{n}}_\perp + f\nabla\eta \cdot \hat{\mathbf{n}}_\| = 0$$

which becomes with $\hat{\mathbf{n}}_\perp = (1, 0)$ and $\hat{\mathbf{n}}_\| = (0, 1)$ at $x = 0$ of

$$\eta_{xt} + f\eta_y = 0.$$

Now if we consider an incident and reflected wave, $\eta = \eta_I + \eta_R$,

$$\eta_I = \hat{\eta}_I \exp[\mathrm{i}(k_I x + l_I y - \omega_I t)] \tag{16.12}$$

$$\eta_R = \hat{\eta}_R \exp[\mathrm{i}(k_R x + l_R y - \omega_R t)] \tag{16.13}$$

Then the boundary condition becomes

$$(k_I\omega_I + f\mathrm{i}l_I)\hat{\eta}_I \exp[\mathrm{i}(l_I y - \omega_I t)] + (k_R\omega_R + f\mathrm{i}l_R)\hat{\eta}_R \exp[\mathrm{i}(l_R y - \omega_R t)] = 0.$$

As with out other reflection problems, this implies that (1) $\omega_I = \omega_R$, (2) $l_R = l_R$, which in turn from the dispersion relationship implies that $|k_R| = |k_I|$ and thus $k_R = -k_I$, and reflection is specular ($\theta_R = \theta_I$). But wait, there is more. What about the wave amplitudes? With these simplifications we can write the boundary condition as

$$(k_I\omega_I + f\mathrm{i}l_I)\hat{\eta}_I + (-k_I\omega_I + \mathrm{i}fl_I)\hat{\eta}_R = 0$$

$$\hat{\eta}_R = -\frac{-k_I\omega_I + \mathrm{i}fl_I}{k_I\omega_I + \mathrm{i}fl_I}\hat{\eta}_I$$

Thus, that magnitudes of the incident and reflected wave are the same, $|\hat{\eta}_R| = |\hat{\eta}_I|$ but the there is a phase shift because of f. If we write $\hat{\eta}_R = -R\hat{\eta}_I$ where R is a complex number given by

$$R = \frac{-k_I\omega_I + \mathrm{i}fl_I}{k_I\omega_I + \mathrm{i}fl_I},$$

then one can show that $|R| = 1$ and just represents a phase shift of $e^{\mathrm{i}\phi}$. Note that if $f = 0$ then $\hat{\eta}_R = \hat{\eta}_I$ and there is no phase shift.

16.5.5 Channel Modes

16.6 Kelvin Waves

Lord Kelvin showed that with the addition of a boundary there is an additional special kind of solution which we now know as a *Kelvin wave*. Consider a boundary at $x = 0$ where $u = 0$ everywhere. Now imagine a solution that is geostrophic in the cross-shore x momentum equation but is a pure gravity wave in the alongshore y direction, and with continuity the equations are

$$
\begin{aligned}
-fv &= g\eta_x \\
v_t &= -g\eta_y \\
\eta_t + hv_y &= 0.
\end{aligned}
$$

Now let $\eta(x, y, t)$ be separable so that $\eta = N(x)\hat{\eta}(y, t)$. The 2nd and 3rd equation can be converted into a wave equation for $\hat{\eta}(y, t)$,

$$\hat{\eta}_{tt} + gh\hat{\eta}_{yy} = 0$$

which gives a dispersion relationship $\omega^2 = (gh)l^2$ which gives a part of the solution

$$\hat{\eta}(x, y) = \hat{\eta}_0 \exp[\mathrm{i}(ly - \omega t)].$$

We can now examine the cross-shore dependence term $N(x)$. Taking the time derivative of the cross-shore momentum equation and substituting back the alongshore momentum equation yields

$$
\begin{aligned}
-fv_t &= -g\eta_{xt} \\
g(f\eta_y + \eta_{xt}) &= 0
\end{aligned}
$$

and substituting in the $\eta = N(x)\hat{\eta}(y, t)$ solution we get

$$
\begin{aligned}
f\mathrm{i}lN - \mathrm{i}\omega\frac{\mathrm{d}N}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}N}{\mathrm{d}x} - \frac{fl}{\omega}N &= 0 \\
\frac{\mathrm{d}N}{\mathrm{d}x} - \frac{f}{(gh)^{1/2}}N &= 0 \\
\frac{\mathrm{d}N}{\mathrm{d}x} - \frac{1}{L_d}N &= 0
\end{aligned}
$$

where the length-scale $L_d = (gh)^{1/2}/f$ a fundamental quantity called the Rossby deformation radius. In this particular case it is the barotropic Rossby deformation radius (also sometimes called the "external" Rossby deformation radius). Another way of interpreting $L_d = f/(gh)^{1/2}$ is as lf/ω. Thus this has a sign. For waves propagating in the $+y$ direction l/ω is positive. For waves propagating in the $-y$ direction l/ω is negative.

Consider $+y$ propagating waves at a wall at $x = 0$ so that the ocean is at $x \leq 0$. The solution has the form

$$N(x) = e^{x/L_d}$$

so that as $x \to -\infty$, $N \to 0$, and the full solution is

$$\eta(x, y, t) = \hat{\eta}_0 e^{x/Ld} \exp[\mathrm{i}(ly - \omega t)] \tag{16.14}$$

If we consider $-y$ propagating waves, the l/ω is negative and then

$$N(x) = e^{-x/L_d}$$

so that as $x \to -\infty$, $N \to \infty$ and we do not like that solution. Thus, Kelvin waves can only propagate with the coast on the right of the propagation direction in the Northern Hemisphere. If the sign of f changes as in the Southern Hemisphere, then Kelvin waves propagate with the coast on the left of propagation direction.

The full single Kelvin wave solution for $x \leq 0$ is thus

$$\eta(x, y, t) = \hat{\eta}_0 \exp(x/L_d) \exp[\mathrm{i}l(y - c_0 t)]$$

Thus a Kelvin wave propagates alongcoast with coast to the right (for positive f) at phase speed $(gh)^{1/2}$ and an offshore exponential decay in η that decays with length-scale $L_d = (c/f)^{1/2}$. Interestingly, Kelvin waves have no low-frequency limit on their frequency as do Inertial-Gravity waves.

16.7 Kelvin Waves in a Channel

Imagine now that one has an infinite channel running north/south with a boundary at $x = 0$ and $x = L$ so that the channel width is L. Because $u = 0$ for waves propagating north along $x = L$ and wave propagating south at $x = 0$ (in the northern hemisphere), This means we can sum the two Kelvin wave solutions as they satisfy the boundary condition at both $x = 0, L$. Consider a "north" (η_n) and "south" (η_s) of equal amplitude. The solution is

$$\eta(x, y, t) = \eta_\mathrm{n} + \eta_\mathrm{s} = \hat{\eta}_0 \left(\exp(-x/L_d) \cos(l(y - c_0 t) + \exp((x - L)/L_d)) \cos(l(y + c_0 t)) \right),$$

at $x = L/2$ the cross-channel exponential decay terms are the same and the solution can be written there as (CHECK)

$$\eta(x = L/2, y, t) = 2\hat{\eta}_0 \exp(-L/(2L_d)) \cos(ly) \cos(c_0 lt).$$

Thus, these waves have points where the sea-surface is always equal to zero at $y = n\pi + \pi/2$ where n is an integer! This is called an amphidromic point and this is the start of realistic tides.

16.8 Problem Set

1. Show how the linear potential vorticity conservation (16.7) can be derived from the full potential vorticity conservation (16.8) assuming η and $\mathbf{u}$ are "small". Recall that $(1 + \epsilon)^{-1} \approx 1 - \epsilon$.

2. More linear potential vorticity: Consider (i) inertial-gravity waves in an infinite ocean f-plane and (ii) Kelvin waves propagating in $+y$ on a f-plane semi-infinite ocean. For (i) and (ii),

   (a) What is the relative vorticity $\zeta = v_x - u_y$?

   (b) What is the linear potential vorticity (16.7)? Is linear PV conserved?

3. For baroclinic mode-1 inertial-gravity waves with constant f, N, and h. See chapter 14

   (a) What is the dispersion relationship?

   (b) What is the mode-1 internal deformation radius L_d?

   (c) Can the dispersion relationship be expressed as the modal rotating internal wave dispersion relationship? If so how?

4. Using the instantaneous energy conservation for a constant depth h, use the plane wave solutions for a plane wave propagating in x and wave average to show that wave energy conservation holds *i.e.,* $E_t + (c_g E)_x = 0$

5. For an ocean of depth $h = 1000$ m everywhere and a constant $N = 3 \times 10^{-3}$ s^{-1} on an f-plane with $f = 10^{-4}$ s^{-1}.

   (a) What is the barotropic deformation radius?

   (b) How much time does it take for a barotropic Kelvin wave to travel the distance from the equator to California? (Take the approximate straight line distance from Ecuador to Pt. Conception CA with Google Earth).

(c) For the mode-1 baroclinic SWE (see Chapter 14), derive the solution for a mode-1 internal Kelvin wave

(d) What is the cross-coast decay scale?

(e) What is the internal phase speed? Is it dispersive or non-dispersive?

(f) How long does it take for a mode-1 internal Kelvin wave to propagate from the equator to California?

6. (EXTRA CREDIT) Derive (16.11).

Chapter 17

WKB

17.1 Setup: Equation for slowly varying shoaling waves

Now we re-examine the normally-incident shallow-water shoaling surface gravity waves on a slope that was performed using ray theory. Now, we will solve the problem with the WKB method - an asymptotic theory - to obtain an approximation to the full solution. The WKB method is more powerful and general than ray theory, but also more opaque. Here, we will observe that they give the same results.

Here we start from the PDE governing sea-surface elevation η variations,

$$\frac{\partial^2 \eta}{\partial t^2} = g \frac{\partial}{\partial x}\left(h \frac{\partial \eta}{\partial x}\right) \quad , \quad h = h(x)$$

and set

$$\eta(x,t) = \hat{\eta}(x) e^{-i\omega_0 t}$$

since $h(x)$ is time-independent. Then

$$\frac{\mathrm{d}^2\hat{\eta}}{\mathrm{d}x^2} + \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} + \frac{\omega_0^2 \hat{\eta}}{gh} = 0$$

Now, non-dimensionalizing as follows:

$$\hat{\eta} = N\hat{\eta}'$$
$$h = h_0 h'$$
$$x = Lx'$$

(where primes denote dimensionless quantities and $\omega_0^2/k_0^2 = gh_0$). The nondimensional equation becomes (dropping the primes)

$$\epsilon^2 \left[\frac{\mathrm{d}^2\hat{\eta}}{\mathrm{d}x^2} + \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right] + \frac{\hat{\eta}}{h} = 0$$

where $\epsilon = 1/k_0 L = \text{wavelength}/2\pi L \ll 1$ provided that $h(x)$ varies slowly compared to the wavelength of the motion.

The form of the equation suggests an expansion in ϵ, but we expect singular behavior as $\epsilon \to 0$ because e^2 multiplies the highest derivatives.

17.2 Solution with WKB

The WKB method proposes a solution of the form

$$\hat{\eta}(x;\epsilon) = e^{S(x,\epsilon)}$$

where

$$S(x,\epsilon) \sim \frac{S_0(x)}{\epsilon} + S_1(x) + \epsilon S_2(x) + \ldots$$

The tilde symbol denotes an *asymptotic series*, by which is meant a series with the property that any truncation (e.g. $\frac{S_0}{\epsilon} + S_1$) approaches $S(x,\epsilon)$ as $\epsilon \to 0$. (The full series need not converge, and usually doesn't.) The equation now is

$$\epsilon^2 \left[\frac{\mathrm{d}^2 S}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}S}{\mathrm{d}x}\right)^2 + \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\mathrm{d}S}{\mathrm{d}x} \right] + \frac{1}{h} = 0$$

and, inserting the expansion and setting coefficients of powers of ϵ to zero, gives

$$\left(\frac{\mathrm{d}S_0}{\mathrm{d}x}\right)^2 = -\frac{1}{h} \qquad \text{(eikonal eqn.)} \qquad O(1)$$

$$\frac{\mathrm{d}^2 S_0}{\mathrm{d}x^2} + 2\frac{\mathrm{d}S_0}{\mathrm{d}x}\frac{\mathrm{d}S_1}{\mathrm{d}x} + \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\mathrm{d}S_0}{\mathrm{d}x} = 0 \qquad O(\epsilon)$$

$$\frac{\mathrm{d}^2 S_0}{\mathrm{d}x^2} + 2\frac{\mathrm{d}S_0}{\mathrm{d}x}\frac{\mathrm{d}S_2}{\mathrm{d}x} + \left(\frac{\mathrm{d}S_1}{\mathrm{d}x}\right)^2 + \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\mathrm{d}S_1}{\mathrm{d}x} = 0 \qquad O(\epsilon^2)$$

etc.

$$\Longrightarrow \qquad S_0(x) = \pm i \int_0^x \frac{1}{(h(\tilde{x}))^{1/2}}\,\mathrm{d}\tilde{x} + \text{const}$$

$$S_1 = \ln h^{-\frac{1}{4}} + \text{const}$$

The approximation $\hat{\eta} \doteq \exp\left[\frac{S_0}{\epsilon} + S_1\right] \iff \hat{\eta} \doteq \frac{\text{const}}{h^{1/4}} \exp\left[\pm\frac{i}{\epsilon}\int_0^x \frac{1}{\sqrt{h(\tilde{x})}}\,\mathrm{d}\tilde{x}\right]$
or in dimensional form

$$\eta(x,t) \doteq \frac{C_\pm}{h^{1/4}} \exp\left[\pm i k_0 \int_0^x \sqrt{\frac{h_0}{h(\tilde{x})}}\,\mathrm{d}\tilde{x} - i\omega_0 t\right]$$

is of the form

$$\hat{\eta} \doteq \mathcal{A}(x,t)e^{i\theta}$$

with

$$\mathcal{A}(x,t) \propto \frac{1}{h(x)^{1/4}} \quad , \quad \theta(x,t) = k_0 \int_0^x \sqrt{\frac{h_0}{h(\tilde{x})}}\,\mathrm{d}\tilde{x} - \omega_0 t$$

where $C_\pm$ are complex constants to be determined by detailed matching to the incoming wave. The WKB solution predicts

$$\mathcal{E} \propto |\mathcal{A}^2| \propto \frac{1}{\sqrt{h}}$$

$$\text{and} \qquad k = \frac{\partial \theta}{\partial x} = k_0 \sqrt{\frac{h_0}{h}}$$

in agreement with the *Ray Theory*.

The WKB method has the advantages:

1. it can be continued (by including more terms) until desired accuracy is reached.

2. the error can be estimated from the next term in the series

It has the disadvantage that it is more cumbersome than the heuristic theory and that the physics gets hidden inside the formalism. However, WKB can handle problems for which the simpler theory is inadequate. (See Bender & Orszag, Chap. 10)

17.3 WKB solution of the caustic problem

Now we start from the equation

$$\frac{\partial^2 \eta}{\partial t^2} = g\frac{\partial}{\partial x}\left(h\frac{\partial \eta}{\partial x}\right) + gh\frac{\partial^2 \eta}{\partial y^2} \tag{1}$$

and, since h depends only on x, seek solutions in the form

$$\eta(x,y,t) = \hat{\eta}(x)e^{i(l_0 y - \omega_0 t)} \tag{2}$$

where

$$\frac{\mathrm{d}^2\hat{\eta}}{\mathrm{d}x^2} + \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} + \left(\frac{\omega_0^2}{gh} - l_0^2\right)\hat{\eta} = 0 \tag{3}$$

Nondimensionalizing as before gives

$$\epsilon^2\left[\frac{\mathrm{d}^2\hat{\eta}}{\mathrm{d}x^2} + \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right] + \left[\frac{1}{h} - \frac{1}{h_0}\right]\hat{\eta} = 0 \tag{4}$$

where $\epsilon = \frac{1}{L(k_0^2+l_0^2)^{1/2}} \ll 1$ if the length scale L for variation of the medium is long compared to a wavelength.
Here

$$h_c = \frac{k_0^2 + l_0^2}{l_0^2} \tag{5}$$

is the nondimensional critical depth. It is convenient to exchange x for a new coordinate $\tilde{x}$ to be chosen such that the first derivative disappears from (4). Using

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}\tilde{x}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}x^2}\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} + \left(\frac{\mathrm{d}\tilde{x}}{\mathrm{d}x}\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}\tilde{x}^2}$$

in (4) gives

$$\epsilon^2 \left[\left(\frac{\mathrm{d}\tilde{x}}{\mathrm{d}x}\right)^2 \frac{\mathrm{d}^2\hat{\eta}}{\mathrm{d}\tilde{x}^2} + \left\{ \frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}x^2} + \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\mathrm{d}\tilde{x}}{\mathrm{d}x} \right\} \frac{\mathrm{d}\hat{\eta}}{\mathrm{d}\tilde{x}} \right] + \left[\frac{1}{h} - \frac{1}{h_c} \right] \hat{\eta} = 0$$

and we want

$$\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}x^2} + \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\mathrm{d}\tilde{x}}{\mathrm{d}x} = \frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}x}\left(h\frac{\mathrm{d}\tilde{x}}{\mathrm{d}x}\right) = 0$$

The choice

$$\tilde{x} = \int_{x_c}^{x} \frac{1}{h(x')}\,\mathrm{d}x' \tag{6}$$

suffices, where

$$h(x_c) = h_c \tag{7}$$

Our problem now takes the simple form

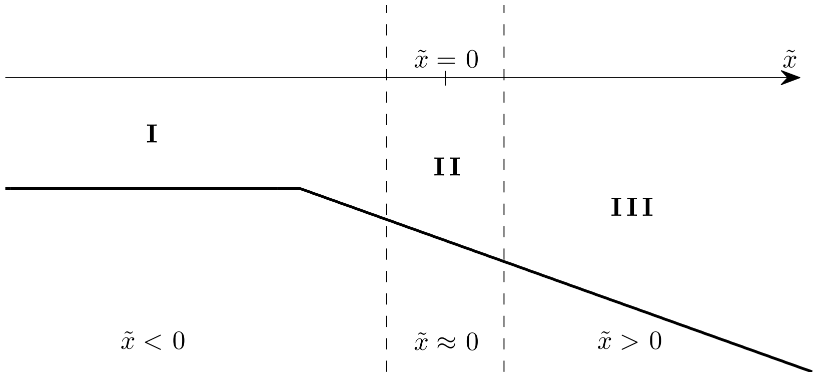
$$\epsilon^2 \frac{\mathrm{d}^2\hat{\eta}}{\mathrm{d}\tilde{x}^2} = Q(\tilde{x})\hat{\eta} \tag{8}$$

where

$$Q(\tilde{x}) = \frac{h}{h_c}(h - h_c) \tag{9}$$

note that $\tilde{x} = 0$ at the caustic, where $h = h_c$ and $Q(\tilde{x}) \gtrless 0$ for $\tilde{x} \gtrless 0$. We thus anticipate oscillatory behavior on $\tilde{x} < 0$ and exponentially decaying behavior for $\tilde{x} > 0$.

It will turn out that the WKB method works only in regions I and III, away from $\tilde{x} = 0$. However, the WKB solutions can be systematically matched to a third asymptotic approximation which is valid in region II. The result will be that we can obtain accurate approximations for all $\tilde{x}$ (all x).



17.3.1 Regions I and III

In regions I and III we again assume

$$\hat{\eta} = e^{S(\tilde{x};\epsilon)}$$

where

$$S(\tilde{x};\epsilon) \sim \frac{S_0(\tilde{x})}{\epsilon} + S_1(\tilde{x}) + \epsilon S_2(\tilde{x}) + \dots \tag{10}$$

Substitution of (10) into (8) gives

$$\left(\frac{\mathrm{d}S_0}{\mathrm{d}\tilde{x}}\right)^2 = Q(\tilde{x}) \qquad O(1)$$

$$\frac{\mathrm{d}^2 S_0}{\mathrm{d}\tilde{x}^2} + 2\frac{\mathrm{d}S_0}{\mathrm{d}\tilde{x}}\frac{\mathrm{d}S_1}{\mathrm{d}\tilde{x}} = 0 \qquad O(\epsilon)$$

$$\frac{\mathrm{d}^2 S_1}{\mathrm{d}\tilde{x}^2} + 2\frac{\mathrm{d}S_0}{\mathrm{d}\tilde{x}}\frac{\mathrm{d}S_2}{\mathrm{d}\tilde{x}} + \left(\frac{\mathrm{d}S_1}{\mathrm{d}\tilde{x}}\right)^2 = 0 \qquad O(\epsilon^2)$$

etc.

with solutions

$$S_0(\tilde{x}) = \pm\int_0^{\tilde{x}} \sqrt{Q(t)}\,\mathrm{d}t$$

$$S_1(\tilde{x}) = -\frac{1}{4}\ln|Q(\tilde{x})|$$

etc.

Since $Q(\tilde{x}) \gtrless 0$ for $\tilde{x} \gtrless 0$, we therefore have

$$\hat{\eta}_{\mathrm{I}} \sim \frac{A}{|Q(\tilde{x})|^{1/4}}\exp\left[\frac{i}{\epsilon}\int_{\tilde{x}}^0 \sqrt{|Q(t)|}\,\mathrm{d}t\right] + \frac{B}{|Q(\tilde{x})|^{1/4}}\exp\left[-\frac{i}{\epsilon}\int_{\tilde{x}}^0 \sqrt{|Q(\tilde{x})|}\,\mathrm{d}t\right] \tag{11}$$

$$\hat{\eta}_{\mathrm{III}} \sim \frac{C}{Q^{1/4}}\exp\left[-\frac{1}{\epsilon}\int_0^{\tilde{x}} \sqrt{Q(t)}\,\mathrm{d}t\right] + \frac{D}{Q^{1/4}}\exp\left[\frac{1}{\epsilon}\int_0^{\tilde{x}} \sqrt{Q(t)}\,\mathrm{d}t\right] \tag{12}$$

where A, B, C, D are constants to be determined.

17.3.2 Regions II

In region II $\tilde{x} \approx 0$ and (8) approximates to

$$\epsilon^2 \frac{\mathrm{d}^2\hat{\eta}}{\mathrm{d}\tilde{x}^2} = a\tilde{x}\hat{\eta} \tag{13}$$

where

$$a = Q'(0)$$

is assumed not to be zero.

We now digress to note that the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = xy(x)$$

is a standard one, with independent solutions $Ai(x)$ and $Bi(x)$, the *Airy functions*. They look like Fig. 17.1 and have the following asymptotic behaviors:

$$Ai(x) \sim \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{3/2}} \quad , \quad x \to \infty$$
$$Bi(x) \sim \frac{1}{\sqrt{\pi}} x^{-\frac{1}{4}} e^{\frac{2}{3}x^{3/2}} \quad , \quad x \to \infty$$
$$Ai(x) \sim \frac{1}{\sqrt{\pi}} |x|^{-\frac{1}{4}} \sin\left[\frac{2}{3}|x|^{\frac{3}{2}} + \frac{\pi}{4}\right] \quad , \quad x \to -\infty$$

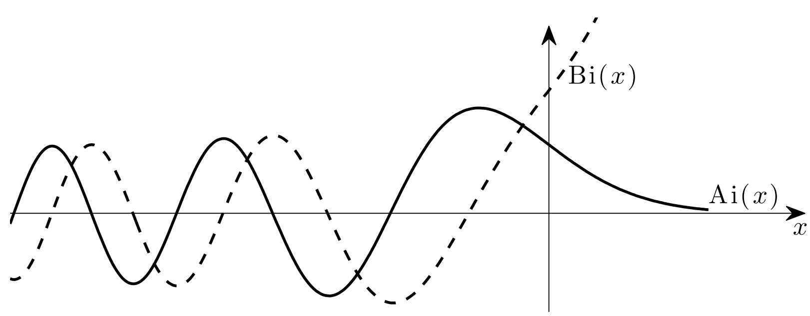


Figure 17.1: Example of Airy Functions

The general solution to (13) is thus

$$\hat{\eta}_{\mathrm{II}} = EAi\left(\frac{a^{\frac{1}{3}}\tilde{x}}{\epsilon^{\frac{2}{3}}}\right) + FBi\left(\frac{a^{\frac{1}{3}}\tilde{x}}{\epsilon^{\frac{2}{3}}}\right)$$

where E and F are two more constants to be determined. (We now have 6 in all.)

The next step is to show that the regions I, II, and III in which these approximations hold overlap. This will allow them to be matched, and eliminate 4 of the undetermined constants. (We should have no more than 2 before applying b.c.'s, because the original equation (8) was second-order.)

Now $\hat{\eta}_{\mathrm{I}}$, $\hat{\eta}_{\mathrm{III}} \to \infty$ as $\tilde{x} \to 0$ so that the WKB approximations evidently break down at the caustic. However, it is possible to determine the regions of their validity as follows: The approximation

$$\hat{\eta} \approx \exp\left(\frac{S_0}{\epsilon} + S_1\right)$$

is valid if $\frac{S_0}{\epsilon} \gg S_1$, $S_1 \gg \epsilon S_2$ and $\epsilon S_2 \ll 1$. (See Bender & Orszag p. 493 - 494 for a discussion.)

Near $\tilde{x} = 0$,

$$\begin{aligned} S_0 &= \pm \int_0^{\tilde{x}} \sqrt{Q(t)}\,\mathrm{d}t \approx \pm \int_0^{\tilde{x}} \sqrt{at} = O\left(a^{\frac{1}{2}} \tilde{x}^{\frac{3}{2}}\right) \quad , \\ S_1 &= -\frac{1}{4} \ln |Q| \approx -\frac{1}{4} \ln a|\tilde{x}| \quad , \\ \text{and} \quad S_2 &\approx \pm \frac{5}{48} a^{-\frac{1}{2}} \tilde{x}^{-\frac{3}{2}} \end{aligned}$$

It can be shown that $S_1 \gg \epsilon S_2$, $\frac{S_0}{\epsilon} \gg S_1$, $\epsilon S_2 \ll 1$ are all satisfied if $|\tilde{x}| \gg \epsilon^{\frac{2}{3}}$. Therefore,

$$\begin{aligned} \hat{\eta} &\approx \hat{\eta}_{\mathrm{I}} \quad \text{for} \quad \tilde{x} \ll -\epsilon^{\frac{2}{3}} \\ \text{and} \quad \hat{\eta} &\approx \hat{\eta}_{\mathrm{III}} \quad \text{for} \quad \tilde{x} \gg \epsilon^{\frac{2}{3}} \end{aligned}$$

The approximations that led to $\hat{\eta}_{\mathrm{II}}$ only required that the Taylor series for $Q(\tilde{x})$ be truncated after the linear term. Thus

$$\hat{\eta} \approx \hat{\eta}_{\mathrm{II}} \quad \text{for} \quad |\tilde{x}| \ll 1$$

For $\epsilon \to 0$, we see that regions I and II, and II and III do indeed overlap.

Next we do the matching and boundary conditions:

First, $D = 0$ in order that $\hat{\eta}_{\mathrm{III}} \to 0$ as $\tilde{x} \to \infty$.

In the overlap region II-III,

$$\hat{\eta}_{\text{III}} \sim \frac{C}{(a\tilde{x})^{1/4}} \exp\left[-\frac{1}{\epsilon}\int_0^{\tilde{x}} \sqrt{at}\,\mathrm{d}t\right] = \frac{C}{(a\tilde{x})^{1/4}} \exp\left[-\frac{2}{3}\frac{a^{\frac{1}{2}}\tilde{x}^{\frac{3}{2}}}{\epsilon}\right] \quad , \quad \tilde{x} \to 0^+$$

and $$\hat{\eta}_{\text{II}} \sim \frac{E}{2\sqrt{\pi}}\frac{\epsilon^{\frac{1}{6}}}{a^{\frac{1}{12}}\tilde{x}^{\frac{1}{4}}} \exp\left[-\frac{2}{3}\frac{a^{\frac{1}{2}}\tilde{x}^{\frac{3}{2}}}{\epsilon}\right] + \frac{F}{\sqrt{\pi}}\frac{\epsilon^{\frac{1}{6}}}{a^{\frac{1}{12}}\tilde{x}^{\frac{1}{4}}} \exp\left[\frac{2}{3}\frac{a^{\frac{1}{2}}\tilde{x}^{\frac{3}{2}}}{\epsilon}\right] \quad , \quad \tilde{x} \to \infty$$

matching requires $F = 0$ and $\frac{C}{a^{\frac{1}{4}}} = \frac{E\epsilon^{\frac{1}{6}}}{2\sqrt{\pi}a^{\frac{1}{12}}}$.

In the overlap region I-II

$$\hat{\eta}_{\text{I}} \sim \frac{A}{a^{\frac{1}{4}}|\tilde{x}|^{\frac{1}{4}}} \exp\left[i\frac{2}{3}\frac{a^{\frac{1}{2}}|\tilde{x}|^{\frac{3}{2}}}{\epsilon}\right] + \frac{B}{a^{\frac{1}{4}}|\tilde{x}|^{\frac{1}{4}}} \exp\left[-i\frac{2}{3}\frac{a^{\frac{1}{2}}|\tilde{x}|^{\frac{3}{2}}}{\epsilon}\right] \quad , \quad \tilde{x} \to 0^-$$

and $$\hat{\eta}_{\text{II}} \sim E\frac{1}{\sqrt{\pi}}\frac{\epsilon^{\frac{1}{6}}}{a^{\frac{1}{12}}|\tilde{x}|^{\frac{1}{4}}} \sin\left[\frac{2}{3}\frac{a^{\frac{1}{2}}|\tilde{x}|^{\frac{3}{2}}}{\epsilon} + \frac{\pi}{4}\right] \quad , \quad \tilde{x} \to -\infty$$

matching requires

$$A = B^* = \frac{E\epsilon^{\frac{1}{6}}a^{\frac{1}{6}}}{2\sqrt{\pi}}e^{-i\frac{\pi}{4}}$$

Collecting all of our results,

$$\hat{\eta}_{\text{I}} = \frac{G}{|Q(\tilde{x})|^{\frac{1}{4}}} \sin\left(\int_{\tilde{x}}^0 \frac{1}{\epsilon}\sqrt{|Q(t)|}\,\mathrm{d}t + \frac{\pi}{4}\right)$$

$$\hat{\eta}_{\text{II}} = \frac{\sqrt{\pi}G}{a^{\frac{1}{6}}\epsilon^{\frac{1}{6}}} Ai\left(\frac{a^{\frac{1}{3}}\tilde{x}}{\epsilon^{\frac{2}{3}}}\right)$$

$$\hat{\eta}_{\text{III}} = \frac{G}{2Q(\tilde{x})^{\frac{1}{4}}} \exp\left[-\frac{1}{\epsilon}\int_0^{\tilde{x}} \sqrt{Q(t)}\,\mathrm{d}t\right]$$

where G is a constant chosen to match the specified amplitude at $\tilde{x} = -\infty$. Further details of the above solution are given in Bender & Orszag (pp. 504 - 510).

The solutions can easily be converted into dimensional form. For example

$$\hat{\eta}_{\text{II}_{\text{dimensional}}} \propto Ai\left((k_0^2 + l_0^2)^{\frac{1}{3}}\left(\frac{\mathrm{d}h}{\mathrm{d}x_c}\frac{1}{h_c}\right)^{\frac{1}{3}}\frac{h_0}{h_c}(x - x_0)\right)$$

but they are in fact easier to interpret in their nondimensional form. Remember that the nondimensional scale for variation of the medium (h) is $O(1)$ and the incident wavelength is $\epsilon \ll 1$. The solutions confirm that the only steady waves possible are standing in x, and they determine the positions of the nodes and anti-nodes relative to the caustic.

The energy of the wave is enhanced by a factor of $\epsilon^{-\frac{1}{3}}$ within a narrow region of width $\epsilon^{\frac{2}{3}}$ near the caustic,

$$\hat{\eta}^2 \propto \frac{1}{a^{\frac{1}{3}}\epsilon^{\frac{1}{3}}} \quad \text{for} \quad |\tilde{x}| < \epsilon^{\frac{2}{3}}$$

This <u>finite peak</u> in energy density corresponds to the <u>infinite</u> energy density predicted by the heuristic theory. It is as if the ray theory of energy propagation remained valid to within $\epsilon^{\frac{2}{3}}$ of the caustic, where that theory would predict $\mathcal{E} \sim \epsilon^{-\frac{1}{3}}$.

It is remarkable that most of these results can be obtained without formal asymptotic analysis and matching. For example, either (1) ray theory, (2) the WKB solution away from $\tilde{x} = 0$ or (3) inspection of the equation

$$\epsilon^2 \frac{\mathrm{d}^2\hat{\eta}}{\mathrm{d}\tilde{x}^2} = Q(\tilde{x})\hat{\eta}$$

will predict the width $\left(|\tilde{x}| = \epsilon^{\frac{2}{3}}\right)$ of the energy peak from the simple observation that the local wavelength $\frac{\epsilon}{\sqrt{Q}} = \frac{\epsilon}{a^{\frac{1}{2}}\tilde{x}^{\frac{1}{2}}}$ becomes large compared to the distance over which Q varies by a sizable fraction of itself, namely $\tilde{x}$, for $|\tilde{x}|$ smaller than $\epsilon^{\frac{2}{3}}$.

The WKB envelope $\hat{\eta}^2 \propto \frac{1}{\sqrt{Q}} \propto \frac{1}{\tilde{x}^{\frac{1}{2}}}$ controls the size of the oscillations until $|\tilde{x}| = \epsilon^{\frac{2}{3}}$, the boundary of the caustic region. Thus $\hat{\eta}^2 \propto \frac{1}{(\epsilon^{\frac{2}{3}})^{\frac{1}{2}}} = \frac{1}{\epsilon^{\frac{1}{3}}}$ in this region. The smaller ϵ gets, the closer the oscillatory behavior gets to $\tilde{x} = 0$ before the Airy function puts a limit on the amplitude, i.e. the farther the oscillatory curve can "ride up" the ascending WKB envelope.

Finally, it will be recalled that $\hat{\eta} \approx \hat{\eta}_{\mathrm{II}}$ over an $O(1)$ region near the caustic, essentially because the only approximation involved in (13) was truncation of the Taylor series. In view of this, it is not too surprising that there exists a <u>single uniformly valid</u> approximate solution. It turns out to be

$$\hat{\eta} \propto \left(\frac{S_0}{\epsilon}\right)^{\frac{1}{6}} \frac{1}{|Q|^{\frac{1}{4}}} Ai\left[\left(\frac{3S_0}{2\epsilon}\right)^{\frac{2}{3}}\right]$$

For more details, see Bender & Orszag page 510.

Chapter 18

Action Conservation

The Wave-Energy Equation

Suppose that the solution to a slowly varying waves problem can be expressed as the superposition of wave packets, each with the same amount of energy. Each wavepacket is characterized by a location $\mathbf{x}(t)$ and a "carrier" wavenumber $\mathbf{k}(t)$. Thus each wavepacket corresponds to a point in $\mathbf{x}-\mathbf{k}$ space. The wavepackets are analogs of the phase space points in the previous section. Let

$$\mathcal{N}(\mathbf{k},\mathbf{x},t)\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{x} \tag{4.1}$$

be the number of wavepackets in the volume $\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{x}$ at time t. Then, in complete analogy with the previous section,

$$\frac{\partial \mathcal{N}}{\partial t} + \frac{\partial}{\partial x_i}(\dot{x}_i \mathcal{N}) + \frac{\partial}{\partial k_i}(\dot{k}_i \mathcal{N}) = 0 \tag{4.2}$$

Finally, suppose that each wavepacket conserves its energy, and that the total energy is the sum of the energy in each wavepacket.

Then if

$$\mathcal{E}(\mathbf{k},\mathbf{x},t)\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{x}$$

is the energy in wavenumber $\mathbf{k}$ at location $\mathbf{x}$ and time t,

$$\mathcal{E}(\mathbf{k},\mathbf{x},t) \propto \mathcal{N}(\mathbf{k},\mathbf{x},t)$$

Therefore,

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial x_i}(\dot{x}_i \mathcal{E}) + \frac{\partial}{\partial k_i}(k_i \mathcal{E}) = 0 \tag{4.3}$$

Equation (4.3) is the generalization of a previously derived wave-energy equation to the case of a full spectrum of slowly varying waves. The definition of $\mathcal{E}(\mathbf{k},\mathbf{x})$ obviously requires a scale separation between the longest waves having significant energy, and the length scale for variability of the locally measured wavenumber spectrum. To obtain our previous equation, assume that the spectrum is mono-chromatic, i.e.

$$\mathcal{E}(\mathbf{k}, \mathbf{x}, t) = \delta\left(\mathbf{k} - \mathbf{k}_0(\mathbf{x}, t)\right) \mathcal{E}(\mathbf{x}, t)$$

and apply $\int\int\int_{-\infty}^{\infty} \mathrm{d}\mathbf{k}$ to (4.3). If $\mathcal{E}(\mathbf{k}, \mathbf{x}, t)$ vanishes at $k_i = \pm\infty$, then

$$\frac{\partial}{\partial t}\mathcal{E}(\mathbf{x}, t) + \frac{\partial}{\partial x_i}\left(\int\int\int \mathrm{d}\mathbf{k}\dot{x}_i\mathcal{E}(\mathbf{k}, \mathbf{x}, t)\right) = 0$$

But

$$\begin{aligned}\int\int\int \mathrm{d}\mathbf{k}\dot{x}_i\mathcal{E}(\mathbf{k}, \mathbf{x}, t) &= \int\int\int \mathrm{d}\mathbf{k}\frac{\partial\Omega}{\partial k_i}(\mathbf{k}, \mathbf{x}, t)\delta(\mathbf{k} - \mathbf{k}_0)\mathcal{E}(\mathbf{x}, t) \\ &= C_g(\mathbf{k}_0, \mathbf{x}, t)\mathcal{E}(\mathbf{x}, t)\end{aligned}$$

and thus

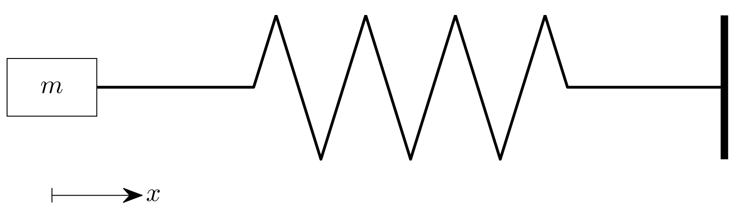
$$\frac{\partial\mathcal{E}}{\partial t} + \frac{\partial}{\partial x_i}(C_{g_i}\mathcal{E}) \tag{4.4}$$

This is a familiar equation.

Conservation of Action

If the medium varies in time, or if mean currents are present, then the energy of individual wavepackets is not conserved, and (4.3) does not hold. Amazingly, however, if the medium and the currents vary slowly, then the individual wavepackets conserve a quantity called "action", and (4.3) can be replaced by a more general equation expressing the conservation of wave action. We begin with the simplest possible example.

Consider a mass on a spring:



If $x = 0$ corresponds to the equilibrium position of the mass, then

$$\ddot{x} = -\omega_0^2 x \tag{5.1}$$

where $\omega_0^2 = \frac{k}{m}$ and k is the spring constant.

Now suppose that either k or m varies slowly in time in a prescribed way. Then $\omega_0(t)$ has a prescribed time variation. By "slow" we mean that the time scale for variation of $\omega_0(t)$ is long compared to the period of the oscillator. An obvious scaling converts (5.1) to the dimensionless form

$$\epsilon^2 \ddot{x} = -\omega_0^2 x$$

for which we seek solutions in the WKB form,

$$x(t) = \exp\left[i\frac{S_0(t)}{\epsilon} + S_1(t) + \dots\right]$$

We easily find that

$$x(t) \sim \frac{A}{\sqrt{\omega_0(t)}} e^{i\int \omega_0(t')dt'}$$

The energy $\mathcal{E}$ averaged over a period of the motion is

$$\mathcal{E}(t) = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} dt\frac{1}{2}m(\dot{x})^2 \propto \omega_0(t)A^2$$

Thus

$$\frac{\mathcal{E}(t)}{\omega_0(t)} \propto A^2 \quad , \quad \text{i.e.} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathcal{E}(t)}{\omega_0(t)}\right)$$

The quantity $\frac{\mathcal{E}}{\omega_0}$ is called action and is conserved by any system composed of slowly varying oscillations. This includes even weakly nonlinear systems in which the slow variations arise from weak couplings between the oscillators.

It turns out that individual wavepackets can be regarded as slowly varying oscillators when viewed in a reference frame which is moving at the velocity of the local mean current. Each wavepacket thus conserves its action,

$$A = \frac{E_r}{\omega_r}$$

where E_r and ω_r are the energy and frequency of the wavepacket relative to the local mean flow. Let

$$A(\mathbf{k}, \mathbf{x}, t)\, \mathrm{d}\mathbf{k}\, \mathrm{d}\mathbf{x}$$

be the total action of all the wavepackets in $\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{x}$ at time t. Then since (4.2) still governs the number density of wavepackets, it follows that

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x_i}(\dot{x}_i A) + \frac{\partial}{\partial k_i}(\dot{k}_i A) = 0$$

is the proper generalization of (4.3) to waves in time-varying media and currents.

When mean currents are present, the ray equations for $\dot{x}_i$ and $\dot{k}_i$ also require modifications. In the presence of currents, the plane-waves dispersion relationship inevitably generalizes to

$$\omega = \mathbf{U}(\mathbf{x}, t) \cdot \mathbf{k} + \Omega(\mathbf{k}, \mathbf{x}, t)$$

where $\mathbf{U}(\mathbf{x}, t)$ is the prescribed mean current and $\mathbf{U} \cdot \mathbf{k}$ represents the Doppler shift. The ray equations then become

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial}{\partial k_i}(\mathbf{U} \cdot \mathbf{k} + \Omega) = U_i + \frac{\partial \Omega}{\partial k_i}$$

$$\frac{\mathrm{d}k_i}{\mathrm{d}t} = -\frac{\partial}{\partial x_i}(\mathbf{U} \cdot \mathbf{k} + \Omega) = -\frac{\partial U_j}{\partial x_i} k_j - \frac{\partial \Omega}{\partial x_i}$$

$$\text{and} \quad \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\partial \mathbf{U}}{\partial t} \cdot \mathbf{k} + \frac{\partial \Omega}{\partial t}$$

If the currents and the medium contain no time dependence, then

$$\omega = \mathbf{U}(\mathbf{x}) \cdot \mathbf{k} + \Omega(\mathbf{k}, \mathbf{x}) = \text{constant}$$

and wavepackets must generally change their relative frequency $\omega_r \equiv \Omega(\mathbf{k}, \mathbf{x})$ to compensate changes in $\mathbf{U} \cdot \mathbf{k}$. However, $\frac{E_r}{\omega_r}$ is conserved. Therefore E_r must change to compensate ω_r. This change in the energy of the wavepacket represents energy lost to, or gained from, the mean flow.

Critical layers occur when wavepackets asymptote to regions where $\omega_r = 0$. As $\omega_r \to 0$, $E_r \to 0$ and the wavepacket energy is irreversibly lost to the mean flow. There are a surprising variety of such phenomena.

Lighthill's derivation of action conservation equation

We will think of internal waves and therefore use the Boussinesq approximation, but the same argument also applies to sound waves. Again, let

$$u_i = \bar{u}_i + u_i'$$

where $\bar{u}_i$ is the mean current and u_i' is the deviation from the mean. Then the linear momentum equation is

$$\rho_0 \left[\frac{\partial u_i'}{\partial t} + \bar{u}_j \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial \bar{u}_i}{\partial x_j} \right] = -\frac{\partial p'}{\partial x_i} + T' \delta_{i3}$$

Now we re-write this equation in an inertial reference frame moving at the velocity of the local mean flow. In this reference frame $\bar{u}_i = 0$ but $\frac{\partial \bar{u}_i}{\partial x_j} \neq 0$. Thus the equation becomes

$$\rho_0 \frac{\partial \bar{u}_i}{\partial t} = -\frac{\partial p'}{\partial x_i} + T'\delta_{i3} - \rho_0 u'_j \frac{\partial \bar{u}_i}{\partial x_j} \tag{1}$$

This equation is the same as would be obtained by linearizing about a state of rest except for the last term. The other equations (continuity and buoyancy) are exactly the same as would be obatined by linearizing about a state of rest.

Next form the energy equation in the comoving reference frame. We must get the same result as for linearizing about a state of rest, except for the contribution of the last term in (1).

$$\frac{\partial}{\partial t}\left(\frac{\rho_0}{2} u'_i u'_i\right) = -\nabla \cdot (\mathbf{u}'p') + \omega' T' - \rho_0 u'_i u'_j \frac{\partial \bar{u}_i}{\partial x_j} \tag{2}$$

Averaging over a wavelength must give

$$\frac{\partial}{\partial t} E_r = -\nabla \cdot \langle \mathbf{u}'p' \rangle - \rho_0 \langle u'_i u'_j \rangle \frac{\partial \bar{u}_i}{\partial x_j} \tag{3}$$

where E_r is the average wave energy measured in this comoving frame. Now if, as in all previous cases,

$$\langle \mathbf{u}'p' \rangle = \mathbf{C}_g E_r \tag{4}$$

then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E_r + \nabla \cdot [\mathbf{C}_g E_r] = -\rho_0 \langle u'_i u'_j \rangle \frac{\partial \bar{u}_i}{\partial x_j} \tag{5}$$

The term on the right-hand side represents the work done on the waves by the mean flow.
Now define:

$$f_i \equiv u'_j \frac{\partial \bar{u}_i}{\partial x_j}$$

and assume that $\bar{u}_i$ has only horizontal components.

Then the only u'_i that contributes to the right-hand side of (5) are also horizontal, and obey the horizontal part of (1), namely

$$\rho_0 \frac{\partial u'_i}{\partial t} = -\frac{\partial p'}{\partial x_i} - \rho_0 f_i \tag{6}$$

If the motions are wave-like, this is

$$-\rho_0 i \omega_r u'_i = -i k_i p' - \rho_0 f_i \tag{7}$$

Thus

$$\rho_0 u_i' = \frac{k_i p'}{\omega_r} - i\frac{\rho_0}{\omega_r} f_i \tag{8}$$

The last term in (8) is 90° out of phase with the first two and hence makes no contribution to

$$-\rho_0 \langle u_i' u_j' \rangle \frac{\partial \bar{u}_i}{\partial x_j} \equiv -\rho_0 \langle u_i' f_i \rangle \tag{9}$$

which therefore becomes

$$\begin{aligned} & -\rho_0 \langle \left(\frac{1}{\rho_0} \frac{k_i}{\omega_r} p' \right) u_j' \rangle \frac{\partial \bar{u}_i}{\partial x_j} \\ = & -\frac{k_i}{\omega_r} C_{g_j} E_r \frac{\partial \bar{u}_i}{\partial x_j} \qquad \text{by (4)} \end{aligned} \tag{10}$$

Thus the wave activity equation becomes

$$\frac{\partial}{\partial t} E_r + \frac{\partial}{\partial x_j}(F_j) = -\frac{k_i}{\omega_r} F_j \frac{\partial \bar{u}_i}{\partial x_j} \tag{11}$$

where $\mathbf{F} = \mathbf{C}_g E_r$. To manipulate (11) into the form of a conservation law, we next develop an equation for ω_r.

We temporarily bo back to the original set of coordinates, in which the mean flow is not locally zero. Suppose that the medium and the mean flow $\bar{\mathbf{u}}$ are steady. Then we know that

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = 0 \qquad \text{along a ray} \tag{12}$$

since

$$\omega = \bar{\mathbf{u}} \cdot \mathbf{k} + \omega_r \tag{13}$$

this means that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\omega_r &= -\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\mathbf{u}} \cdot \mathbf{k}) \\ &= -\bar{u}_i \frac{\mathrm{d}k_i}{\mathrm{d}t} - k_i \frac{\mathrm{d}\bar{u}_i}{\mathrm{d}t} \\ &= -\bar{u}_i \left[-k_j \frac{\partial \bar{u}_j}{\partial x_i} - \frac{\partial \omega_r}{\partial x_i} \right] - k_i \left[\frac{\partial \bar{u}_i}{\partial x_j} \frac{\mathrm{d}x_j}{\mathrm{d}t} + \cancelto{0}{\frac{\partial \bar{u}_i}{\partial t}} \right] \end{aligned} \tag{14}$$

Then substituting

$$\frac{\mathrm{d}x_j}{\mathrm{d}t} = \bar{u}_j + C_{g_j} \tag{15}$$

and noting a cancellation, we have

$$\frac{\mathrm{d}\omega_r}{\mathrm{d}t} = \bar{u}_i \frac{\partial \omega_r}{\partial x_i} - k_i \frac{\partial \bar{u}_i}{\partial x_j} C_{g_j} \tag{16}$$

In the comoving reference frame, (16) becomes

$$\frac{\mathrm{d}\omega_r}{\mathrm{d}t} = -k_i \frac{\partial \bar{u}_i}{\partial x_j} C_{g_j} \tag{17}$$

In this same reference frame, (11) is

$$\frac{\mathrm{d}E_r}{\mathrm{d}t} = -E_r \nabla \cdot \mathbf{C}_g - \frac{k_i}{\omega_r} C_{g_i} \frac{\partial \bar{u}_i}{\partial x_j} \frac{E_r}{\omega_r} \tag{18}$$

Then

$$\frac{1}{\omega_r}(18) - \frac{E_r}{\omega_r^2}(17) \qquad \Longrightarrow$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{E_r}{\omega_r}\right) = -\left(\frac{E_r}{\omega_r}\right) \nabla \cdot \mathbf{C}_g$$

which is the desired result.

This derivation of the action conservation equation seems somewhat unmotivated and not completely rigorous. In particular, it relies on (4), which was never generally proved, and might be itself affected by the presence of the mean flow.

The other approach to action conservation relies on a variational formulation of the whole problem. It seems to be very useful as a unified approach to all of slowly varying theory, even including nonlinear effects, and so we consider it next.

Return of the slowly varying oscillator

A <u>function</u> is a rule that assigns one number to another:

$$f = f(x)$$

A <u>functional</u> is a rule that assigns a number to an entire function, over some range of its argument:

$$F = f[x(t)]$$

Functionals are frequently defined by integrals, for example:

$$F[x(t)] = \int_{t_1}^{t_2} \left\{ \frac{1}{2} m \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \frac{1}{2} k(t) x(t)^2 \right\} dt$$

A typical question about functionals is: For what $x(t)$ does $F[x(t)]$ take a stationary value (a maximum, minimum, or critical point)?

More precisely, for what $x(t)$ is $F[x(t)]$ unchanged by small changes $\delta x(t)$ in $x(t)$ which obey

$$\delta x(t_1) = \delta x(t_2) = 0$$

but are otherwise arbitrary?

To answer the question we compute

$$F[x + \delta x] = \int_{t_1}^{t_2} \left\{ \frac{1}{2} m \left(\frac{\mathrm{d}}{\mathrm{d}t}(x + \delta x) \right)^2 - \frac{1}{2} k(t)(x + \delta x)^2 \right\} dt$$

and subtract $F[x]$.

The result is

$$\begin{aligned} \delta F &\equiv F[[x + \delta x] - F[x] \\ &= \int_{t_1}^{t_2} \left\{ m \frac{\mathrm{d}x}{\mathrm{d}t} \frac{\mathrm{d}\delta x}{\mathrm{d}t} - k(t) x \delta x \right\} + O(\delta x)^2 \end{aligned}$$

Integrating by parts, we get

$$0 = \int_{t_1}^{t_2} \left\{ + \frac{\mathrm{d}}{\mathrm{d}t} \left(m \frac{\mathrm{d}x}{\mathrm{d}t} \delta x \right) - m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \delta x - k(t) x \delta x \right\} dt$$

The first term vanishes because $\delta x = 0$ at $t = t_1, t_2$. Thus we are left with:

$$\int_{t_1}^{t_2} \left\{ m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + k(t) x \right\} dx dt = 0$$

Since $\delta x(t)$ is arbitrary, we must have

$$m\ddot{x} = -k(t)x$$

which is the equation for the mass on the slowly varying spring. This equation is said to be equivalent to the variational principle that gave it. The most famous variational principle is Hamilton's principle (of which this is an example).

For any equation, we can seek the corresponding variational principle. If we find it, then we have an elegant way of doing slowly varying theory. The idea is to insert a slowly varying solution into the Lagrangian functional and then to simplify (by averaging) before taking the variations.

Suppose that

$$x(t) = A(t)\cos\theta(t)$$

where $A(t)$ and $\dot{\theta}(t)$ are slowly varying functions (slow compared to $\theta(t)$). Then

$$\begin{aligned}\dot{x}(t) &= \dot{A}(t)\cos\theta(t) - A(t)\dot{\theta}\sin\theta \\ &\approx -A\dot{\theta}\sin\theta\end{aligned}$$

Then

$$\begin{aligned}F &\cong \int_{t_1}^{t_2}\left[\frac{1}{2}mA^2\dot{\theta}^2\sin^2\theta - \frac{1}{2}kA^2\cos^2\theta\right]dt \\ &\approx \frac{m}{4}\int_{t_1}^{t_2}[A^2\dot{\theta}^2 - \omega_0^2A^2]dt\end{aligned}$$

where $\omega_0^2(t) \equiv \frac{k(t)}{m}$ as before. The variations now yield:

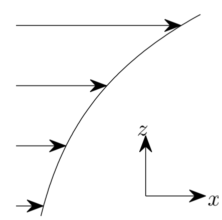
$$\begin{aligned}&\delta A: && \dot{\theta} = \omega_0^2 \\ &\delta\theta: && \frac{\mathrm{d}}{\mathrm{d}t}(A^2\dot{\theta}) = 0\end{aligned}$$

This is the same result as from WKB, but note how action conservation results directly from the variation in phase. The same approach works for wave equations in space and time, and the algebra requred is usually much less than for other types of perturbation theory.

Chapter 19

Sound waves and Internal Waves in a Current

Sound waves and internal gravity waves in a current



Assume that $\mathbf{U} = (\bar{u}(z), 0, 0)$ and that the medium is otherwise uniform. Then the refraction equations tell us that:

$$\frac{\mathrm{d}k}{\mathrm{d}t} = 0 \quad , \quad \frac{\mathrm{d}l}{\mathrm{d}t} \quad , \quad \frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{\mathrm{d}\bar{u}(z)}{\mathrm{d}z}k \quad , \quad \frac{\mathrm{d}\omega}{\mathrm{d}t} = 0$$

Then $k = k_0,\ l = l_0,\ \omega = \omega_0$.
Assume $l = 0$.

Then the unknown $m(z)$ can be determined from the slowly varying dispersion relation alone:

$$\omega + - = k_0\bar{u}(z) + \Omega(k_0, 0, m(z))$$

Sound Waves

In the case of sound waves this is

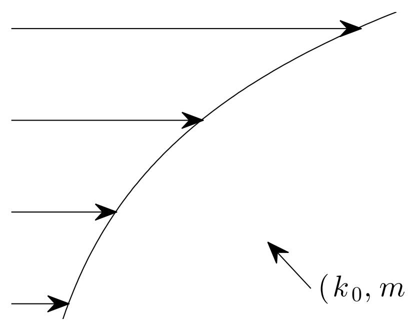
$$\omega_0 = k_0\bar{u}(z) + c\sqrt{k_0^2 + m^2(z)}$$

where c is assumed constant.
The ray equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \bar{u}(z) + \frac{ck_0}{\sqrt{k_0^2 + m^2(z)}} \quad , \quad \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{cm(z)}{\sqrt{k_0^2 m^2(z)}}$$

These three equations are to be solved together. However, the general character of the solutions can be seen without explicitly solving them.

Suppose $\frac{\mathrm{d}\bar{u}}{\mathrm{d}z} > 0$ at all levels and that the waves are incident from below.
Then upwind propagation:



requires that $c\sqrt{k_0^2 + m^2(z)}$ increase to offset the increase in $k_0\bar{u}(z)$. ($k_0 < 0$). As m increases, the rays bend upward.

By the same reasoning, downwind propagation requires that $m(z)$ decrease as $\bar{u}(z)$ increases. A turning point is reached when $m = 0$, i.e. where

$$\omega_0 = k_0\bar{u}(z) + ck_0$$

This is a caustic with the same general properties as in the case of long gravity waves studied in a previous lecture.

The new feature is an energy exchange between the waves and the mean flow. A wavepacket, conserving $\frac{\mathcal{E}_r}{\omega_r}$, gains $\mathcal{E}_r$ as ω_r increases in upstream propagation. Conversely, $\mathcal{E}_r$ decreases as ω_r decreases in downstream propagation.

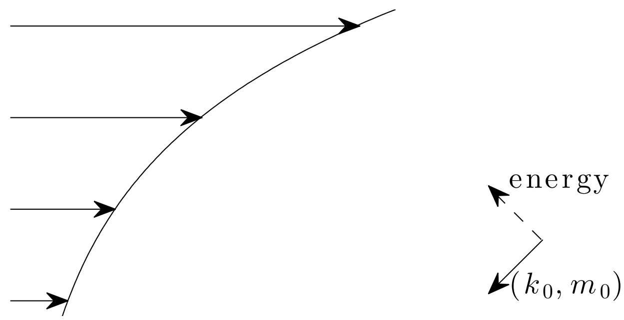
These energy changes are absolute, and distinct from the energy density changes that arise from ray tube convergence near the caustic. However, both effects are reversible as rays are reflected at the caustic: the energy lost to the current is regained as ω_r increases again in the reflected wave.

Internal waves

The situation for internal waves is much more complex. The slowly varying dispersion relation is now

$$\omega_0 = k_0\bar{u}(z) + \frac{N|k_0|}{\sqrt{k_0^2 + m^2(z)}}$$

For upstream energy propagation



$k_0 < 0$, and $|m|$ must decrease as $\bar{u}(z)$ increases. A turning point is reached when $m - 0$ ($\omega_r = N$). To be sure of what happens there, we must examine the behavior of the ray equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \bar{u}(z) - \frac{Nm^2}{(k^2+m^2)^{\frac{3}{2}}} \quad , \quad \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{-N|k|m}{(k^2+m^2)^{\frac{3}{2}}}$$

near the turning point, where $\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right) = (\bar{u}, 0)$.

Let $z = 0$ correspond to the turning point. Then the dispersion relation

$$\omega_0 - k_0\bar{u}(z) \approx 1 - \frac{1}{2}\frac{Nm^2(z)}{k_0^2}$$

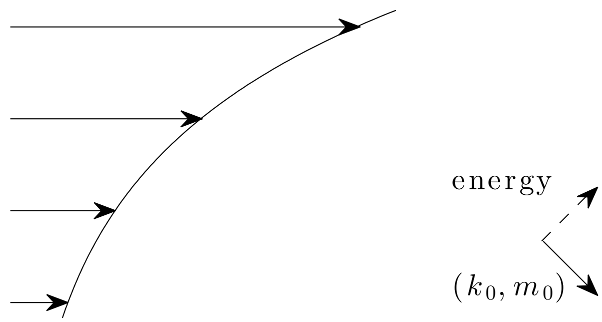
tells us that $|m| \sim \sqrt{|z|}$ near the turning point. Thus

$$\frac{\mathrm{d}z}{\mathrm{d}t} \approx -Nm \sim \sqrt{z}$$

$$\Longrightarrow \qquad \int \frac{\mathrm{d}z}{\sqrt{z}} = \int \mathrm{d}t$$

Since $\frac{1}{\sqrt{z}}$ has an integrable singularity, the ray reaches the turning point in a finite time, and turns back in a qualitatively similar way to sound waves.

Internal waves propagating downstream:



behave quite differently. Since $k_0 > 0$ now, $m(z)$ must increase as $\bar{u}(z)$ increases. However, ω_r can get no smaller than zero. A critical level is reached when

$$\omega_0 = k_0 \bar{u}(z) + 0$$

i.e. where the x-direction phase velocity $\frac{\omega_0}{k_0}$ equals $\bar{u}(z)$. The ray equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \bar{u}(z) + \frac{Nm^2}{(k^2+m^2)^{\frac{3}{2}}} \quad , \quad \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{-Nkm}{(k^2+m^2)^{\frac{3}{2}}}$$

Again $\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right) = (\bar{u}, 0)$ at the critical level, but the behavior near the critical level is quite different than before.

Let $z = 0$ correspond to the critical level. Then the dispersion relation

$$\omega_0 = k_0 \bar{u}(z) \approx \frac{Nk_0}{m(z)}$$

tells us that

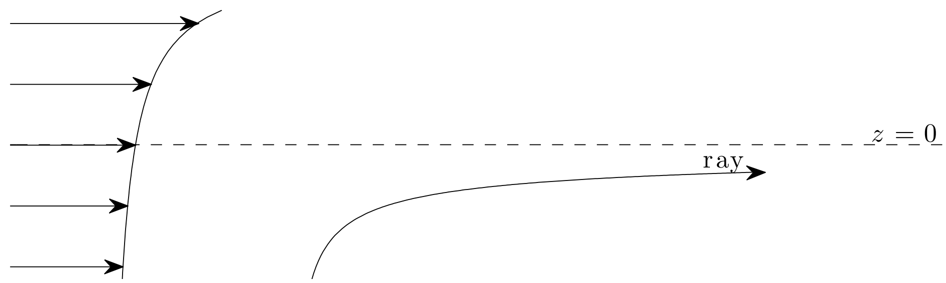
$$m(z) \sim \frac{1}{\sqrt{z}}$$

near the critical level. Thus $\frac{\mathrm{d}z}{\mathrm{d}t} \sim \frac{1}{m^2} \sim z^2$ near the critical level, and

$$\int \frac{\mathrm{d}z}{z^2} = \int \mathrm{d}t$$

implies that rays reach the critical level only after an infinite time. In other words, the ray is an asymptote.

Since $\omega_r \to 0$ as the ray approaches the critical level, action conservation demands that $\mathcal{E}_r \to 0$. Therefore, wavepackets lose all of their energy to the mean current flow as the critical layer is approaches.



Technically, ray theory breaks down at both turning points and critical levels. Asymptotic matching theory fixes the problem at the turning point (in a manner similar to the Airy-function analysis in a previous lecture). However, asymptotic analysis is not sufficient for a close analysis of the critical level; the solution of the differential equation turns out to have a branch singularity there. What is really needed is additional physics. One must add one of the following:

1.) viscosity

2.) nonlinearity

3.) or reconsider as an initial value problem (See Booker & Bretherton, op. cit.)

All of this assumes that the mean flow is stable. If not, other phenomena (e.g. over-reflection) can occur.

Chapter 20

Appendix A: The Analogy Between Ray Theory and Hamiltonian Mechanics

This appendix outlines the analogy between ray theory and mechanics. This analogy can be carried rather far, and it illuminates both fields.

We begin with a single-paragraph summary of ray theory: Given a linear waves problem, we can seek a slowly-varying solution in the form

$$\psi \sim A(\mathbf{x},t)e^{i\theta(\mathbf{x},t)} \quad , \tag{1.1}$$

where the wavevector,

$$\mathbf{k}(\mathbf{x},t) \equiv \frac{\partial\theta}{\partial\mathbf{x}} \tag{1.2}$$

and frequency

$$\omega(\mathbf{x},t) \equiv -\frac{\partial\theta}{\partial t} \tag{1.3}$$

are assumed to be large compared to the inverse length (and time) scales for the variability of A, k, and ω.

Let

$$\omega = \Omega(k,D) \tag{1.4}$$

be the dispersion relation for the plane waves in the special case of a constant medium. (Again, D is a general medium parameter.) Then, for the slowly varying wave, it is reasonable to require that:

$$\omega(\mathbf{x},t) = \Omega\left(\mathbf{k}(\mathbf{x},t), D(\mathbf{x},t)\right) \equiv \Omega(\mathbf{k},\mathbf{x},t) \tag{1.5}$$

i.e., that the slowly varying frequency and wavenumber obey the local plane wave dispersion relationship. Everything follows from (1.2, 1.3, 1.5).

The problem defined by (1.2, 1.3, 1.5) can be posed in several equivalent ways. For example, we can use (1.2) and (1.3) to eliminate $\mathbf{k}$ and ω, and write (1.5) in the form

$$\frac{\partial\theta}{\partial t} + \Omega\left(\frac{\partial\theta}{\partial\mathbf{x}}, \mathbf{x}, t\right) = 0 \tag{1.6}$$

Alternatively, we can write (1.2, 1.3, 1.5) as "ray equations", viz.

$$\boxed{\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\partial\Omega}{\partial\mathbf{k}} \quad , \quad \frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t} = -\frac{\mathrm{d}\Omega}{\mathrm{d}\mathbf{x}}} \quad , \quad \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\partial\Omega}{\partial t} \tag{1.7}$$

These equations were derived in the previous section. As explained there, the first two of (1.7) can be solved independently of the last.

The Mechanical Analogy

Every mechanical system may be described by Hamilton's canonical equations. For example, let $\mathbf{q}$ be the location of a particle with mass m moving in a force potential $V(\mathbf{q},t)$. If we define

$$\mathbf{p} \equiv m\dot{\mathbf{q}} \quad \text{and} \quad H(\mathbf{p},\mathbf{q},t) \equiv \frac{1}{2m}\mathbf{p}\cdot\mathbf{p} + V(\mathbf{q},t)$$

then Newton's law,

$$m\ddot{\mathbf{q}} = -\frac{\partial V}{\partial\mathbf{q}}$$

is equivalent to the canonical equations:

$$\boxed{\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial H}{\partial\mathbf{p}} \quad , \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial\mathbf{q}}} \quad , \quad \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial H}{\partial t} \tag{1.8}$$

where E is the energy of the particle.

Equations (1.7) and (1.8) are analogous. The analogy is between:

$$\mathbf{x} \longleftrightarrow \mathbf{q}$$

$$\mathbf{k} \longleftrightarrow \mathbf{p}$$

$$\omega(\mathbf{x},t) \longleftrightarrow E(\mathbf{q},t)$$

$$\Omega(\mathbf{k},\mathbf{x},t) \longleftrightarrow H(\mathbf{p},\mathbf{q},t)$$

Thus every mechanical problem corresponds to a waves problem and vice-versa.

Mechanical problems are frequently solved using the canonical equations (1.8). However, we may instead use the analog of (1.6) viz.

$$\frac{\mathrm{d}S}{\mathrm{d}t} + H\left(\frac{\partial S}{\partial \mathbf{q}}, \mathbf{q}, t\right) = 0, \tag{1.9}$$

which is called the Hamilton-Jacobi equation. Here $S(\mathbf{q}, t)$, which is the analogy of $\theta(\mathbf{x}, t)$, is defined by the equations

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}} \quad \text{and} \quad E = -\frac{\partial S}{\partial t}, \tag{1.10}$$

which are analogues of (1.2) and (1.3). $S(\mathbf{q}, t)$ is called Hamilton's principal function.

The discussion of these equations in mechanics books is sometimes made very abstract, but once the complete analogy with the waves problem is realized, the entire development is almost trivial.

This analogy can be exploited in other ways, too. For example, it is well known that the canonical equations (1.8) are equivalent to a variational principle (actually several). It follows that the equations for slowly varying waves are also equivalent to a variational principle. The most famous variational principle for waves is Fermat's principle. The mechanical analogy of Fermat's principle is Jacobi's principle. Historically, Fermat's principle came first.

Connection with Quantum Mechanics

The analogy between classical mechanics and the equations for slowly varying waves is so remarkable that it invites the following conjecture: that Hamilton's equations are not merely analogous to, but actually are, slowly varying approximations to an underlying, more fundamental wave equation. The most obvious difficulty with this conjecture is that the dimensions of the momentum $\mathbf{p}$ are different from the dimensions of the wavenumber $\mathbf{k}$. But suppose that the momentum $\mathbf{p}$ and energy E are related to the wavenumber $\mathbf{k}$ and frequency ω by

$$\mathbf{p} = \bar{h}\mathbf{k} \quad \text{and} \quad E = \bar{h}\omega$$

If the constant $\bar{h}$ (Planck's constant) has dimensions ML^2/T, then (with $\mathbf{q} = \mathbf{x}$), Hamilton's equations (1.8) are equivalent to

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial \Omega}{\partial k_i} \quad , \quad \frac{\mathrm{d}k_i}{\mathrm{d}t} = -\frac{\partial \Omega}{\partial x_i} \quad , \quad \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\partial \Omega}{\partial t} \tag{1.7}$$

where

$$\Omega(\mathbf{k}, \mathbf{x}, t) = \frac{1}{\bar{h}} H(\bar{h}\mathbf{k}, \mathbf{x}, t) \tag{2.1}$$

The form of the underlying wave equation can now be deduced from the dispersion relation

$$\omega = \frac{1}{\bar{h}} H(\bar{h}\mathbf{k}, \mathbf{x}, t) \tag{2.2}$$

by simply reversing the procedure by which the dispersion relation is obtained from the original wave equation. This amounts to the replacements

$$-i\omega \to \frac{\partial}{\partial t} \quad , \quad i\mathbf{k} \to \nabla$$

Using

$$H(\mathbf{p}, \mathbf{q}, t) \equiv \frac{1}{2m}\mathbf{p} \cdot \mathbf{p} + V(\mathbf{q}, t),$$

we easily obtain Schrödinger's equation,

$$i\bar{h}\frac{\partial \psi}{\partial t} = \frac{1}{2m}\left[-\bar{h}^2\nabla^2 + V\right]\psi \tag{2.3}$$

This method of "deriving" quantum mechanics makes the correspondence between operators and classical variables seem far less mysterious than do other ways of introducing the subject.

Liouville's Equation

Liouville's equation is the basis for statistical mechanics. It turns out that it is also the mechanical analog of the "energy equation" for the waves system. We return to the canonical equations for a classical particle:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad , \quad i = 1, 2, 3 \tag{3.1}$$

The six-dimensional space spanned by $(q_1, q_2, q_3, p_1, p_2, p_3)$ is called phase space. Every point in phase space represents a possible position-momentum state of the particle. Every realization of (3.1) is a trajectory in phase space.

Consider a collection of moving points in phase space. Let

$$\mathcal{P}(\mathbf{q}, \mathbf{p}, t)\, \mathrm{d}\mathbf{q}\, \mathrm{d}\mathbf{p} \tag{3.2}$$

be the number of points in the phase volume $\mathrm{d}\mathbf{q}\, \mathrm{d}\mathbf{p}$ at time t. Since these points are neither created nor destroyed,

$$\frac{\partial \mathcal{P}}{\partial t} + \frac{\partial}{\partial q_i}(\dot{q}_i \mathcal{P}) + \frac{\partial}{\partial p_i}(\dot{p}_i \mathcal{P}) = 0 \tag{3.3}$$

This equation is the analogy of the continuity equation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (\mathbf{u}\rho) = 0$$

expressing the conservation of mass-points in ordinary three-dimensional space. It follows from the canonical equations (3.1) that

$$\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} = 0 \tag{3.4}$$

and thus (3.3) may be written

$$\frac{\partial \mathcal{P}}{\partial t} + \dot{q}_i \frac{\partial \mathcal{P}}{\partial q_i} + \dot{p}_i \frac{\partial \mathcal{P}}{\partial p_i} = 0$$

which is the analog of the continuity equation for an incompressible fluid.

The Wave-Energy Equation

Suppose that the solution to a slowly varying waves problem can be expressed as the superposition of wave packets, each with the same amount of energy. Each wavepacket is characterized by a location $\mathbf{x}(t)$ and a "carrier" wavenumber $\mathbf{k}(t)$. Thus each wavepacket corresponds to a point in $\mathbf{x}-\mathbf{k}$ space. The wavepackets are analogs of the phase space points in the previous section. Let

$$\mathcal{N}(\mathbf{k}, \mathbf{x}, t)\, \mathrm{d}\mathbf{k}\, \mathrm{d}\mathbf{x} \tag{4.1}$$

be the number of wavepackets in the volume $\mathrm{d}\mathbf{k}\, \mathrm{d}\mathbf{x}$ at time t. Then, in complete analogy with the previous section,

$$\frac{\partial \mathcal{N}}{\partial t} + \frac{\partial}{\partial x_i}(\dot{x}_i \mathcal{N}) + \frac{\partial}{\partial k_i}(\dot{k}_i \mathcal{N}) = 0 \tag{4.2}$$

Finally, suppose that each wavepacket conserves its energy, and that the total energy is the sum of the energy in each wavepacket.

Then if

$$\mathcal{E}(\mathbf{k}, \mathbf{x}, t)\, \mathrm{d}\mathbf{k}\, \mathrm{d}\mathbf{x}$$

is the energy in wavenumber $\mathbf{k}$ at location $\mathbf{x}$ and time t,

$$\mathcal{E}(\mathbf{k}, \mathbf{x}, t) \propto \mathcal{N}(\mathbf{k}, \mathbf{x}, t)$$

Therefore,

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial x_i}(\dot{x}_i \mathcal{E}) + \frac{\partial}{\partial k_i}(k_i \mathcal{E}) = 0 \tag{4.3}$$

Equation (4.3) is the generalization of a previously derived wave-energy equation to the case of a full spectrum of slowly varying waves. The definition of $\mathcal{E}(\mathbf{k}, \mathbf{x})$ obviously requires a scale separation between the longest waves having significant energy, and the length scale for variability of the locally measured wavenumber spectrum. To obtain our previous equation, assume that the spectrum is mono-chromatic, i.e.

$$\mathcal{E}(\mathbf{k}, \mathbf{x}, t) = \delta\left(\mathbf{k} - \mathbf{k}_0(\mathbf{x}, t)\right) \mathcal{E}(\mathbf{x}, t)$$

and apply $\int\int\int_{-\infty}^{\infty} \mathrm{d}\mathbf{k}$ to (4.3). If $\mathcal{E}(\mathbf{k}, \mathbf{x}, t)$ vanishes at $k_i = \pm\infty$, then

$$\frac{\partial}{\partial t}\mathcal{E}(\mathbf{x}, t) + \frac{\partial}{\partial x_i}\left(\int\int\int \mathrm{d}\mathbf{k} \dot{x}_i \mathcal{E}(\mathbf{k}, \mathbf{x}, t)\right) = 0$$

But

$$\begin{aligned} \int\int\int \mathrm{d}\mathbf{k} \dot{x}_i \mathcal{E}(\mathbf{k}, \mathbf{x}, t) &= \int\int\int \mathrm{d}\mathbf{k} \frac{\partial \Omega}{\partial k_i}(\mathbf{k}, \mathbf{x}, t) \delta(\mathbf{k} - \mathbf{k}_0) \mathcal{E}(\mathbf{x}, t) \\ &= C_g(\mathbf{k}_0, \mathbf{x}, t) \mathcal{E}(\mathbf{x}, t) \end{aligned}$$

and thus

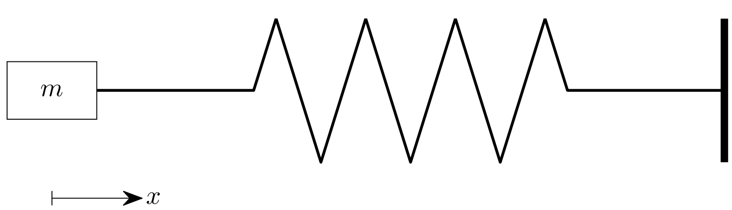
$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial x_i}(C_{g_i} \mathcal{E}) \tag{4.4}$$

This is a familiar equation.

Conservation of Action

If the medium varies in time, or if mean currents are present, then the energy of individual wavepackets is not conserved, and (4.3) does not hold. Amazingly, however, if the medium and the currents vary slowly, then the individual wavepackets conserve a quantity called "action", and (4.3) can be replaced by a more general equation expressing the conservation of wave action. We begin with the simplest possible example.

Consider a mass on a spring:



If $x = 0$ corresponds to the equilibrium position of the mass, then

$$\ddot{x} = -\omega_0^2 x \tag{5.1}$$

where $\omega_0^2 = \frac{k}{m}$ and k is the spring constant.

Now suppose that either k or m varies <u>slowly</u> in time in a prescribed way. Then $\omega_0(t)$ has a prescribed time variation. By "slow" we mean that the time scale for variation of $\omega_0(t)$ is long compared to the period of the oscillator. An obvious scaling converts (5.1) to the dimensionless form

$$\epsilon^2 \ddot{x} = -\omega_0^2 x$$

for which we seek solutions in the WKB form,

$$x(t) = \exp\left[i\frac{S_0(t)}{\epsilon} + S_1(t) + \ldots\right]$$

We easily find that

$$x(t) \sim \frac{A}{\sqrt{\omega_0(t)}} e^{i\int \omega_0(t')dt'}$$

The energy $\mathcal{E}$ averaged over a period of the motion is

$$\mathcal{E}(t) = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} dt \frac{1}{2} m(\dot{x})^2 \propto \omega_0(t) A^2$$

Thus

$$\frac{\mathcal{E}(t)}{\omega_0(t)} \propto A^2 \quad , \quad \text{i.e.} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathcal{E}(t)}{\omega_0(t)}\right)$$

The quantity $\frac{\mathcal{E}}{\omega_0}$ is called <u>action</u> and is conserved by any system composed of slowly varying oscillations. This includes even weakly <u>nonlinear</u> systems in which the slow variations arise from weak couplings between the oscillators.

It turns out that individual wavepackets can be regarded as slowly varying oscillators when viewed in a reference frame which is moving at the velocity of the local mean current. Each wavepacket thus conserves its action,

$$A = \frac{E_r}{\omega_r}$$

where E_r and ω_r are the energy and frequency of the wavepacket <u>relative</u> to the <u>local mean flow</u>. Let

$$A(\mathbf{k}, \mathbf{x}, t)\, \mathrm{d}\mathbf{k}\, \mathrm{d}\mathbf{x}$$

be the total action of all the wavepackets in $d\mathbf{k}\,d\mathbf{x}$ at time t. Then since (4.2) still governs the number density of wavepackets, it follows that

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x_i}(\dot{x}_i A) + \frac{\partial}{\partial k_i}(\dot{k}_i A) = 0$$

is the proper generalization of (4.3) to waves in time-varying media and currents.

When mean currents are present, the ray equations for $\dot{x}_i$ and $\dot{k}_i$ also require modifications. In the presence of currents, the plane-waves dispersion relationship inevitably generalizes to

$$\omega = \mathbf{U}(\mathbf{x}, t) \cdot \mathbf{k} + \Omega(\mathbf{k}, \mathbf{x}, t)$$

where $\mathbf{U}(\mathbf{x}, t)$ is the prescribed mean current and $\mathbf{U} \cdot \mathbf{k}$ represents the Doppler shift. The ray equations then become

$$\frac{dx_i}{dt} = \frac{\partial}{\partial k_i}(\mathbf{U} \cdot \mathbf{k} + \Omega) = U_i + \frac{\partial \Omega}{\partial k_i}$$

$$\frac{dk_i}{dt} = -\frac{\partial}{\partial x_i}(\mathbf{U} \cdot \mathbf{k} + \Omega) = -\frac{\partial U_j}{\partial x_i} k_j - \frac{\partial \Omega}{\partial x_i}$$

and $$\frac{d\omega}{dt} = \frac{\partial \mathbf{U}}{\partial t} \cdot \mathbf{k} + \frac{\partial \Omega}{\partial t}$$

If the currents and the medium contain no time dependence, then

$$\omega = \mathbf{U}(\mathbf{x}) \cdot \mathbf{k} + \Omega(\mathbf{k}, \mathbf{x}) = \text{constant}$$

and wavepackets must generally change their relative frequency $\omega_r \equiv \Omega(\mathbf{k}, \mathbf{x})$ to compensate changes in $\mathbf{U} \cdot \mathbf{k}$. However, $\frac{E_r}{\omega_r}$ is conserved. Therefore E_r must change to compensate ω_r. This change in the energy of the wavepacket represents energy lost to, or gained from, the mean flow.

Critical layers occur when wavepackets asymptote to regions where $\omega_r = 0$. As $\omega_r \to 0$, $E_r \to 0$ and the wavepacket energy is irreversibly lost to the mean flow. There are a surprising variety of such phenomena.

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Chapter 21

Currents Effects on Waves

21.1 Nonlinear shallow water equations

The nonlinear shallow water equations are in 2 dimensions (momentum and continuity)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial \eta}{\partial t} + \frac{\partial[(\eta + h)u]}{\partial x} = 0$$

21.2 Review of Standard Linearization

If we linearize on a background current of $\mathbf{u} = 0$, then we get,

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial \eta}{\partial t} + \frac{\partial(hu)}{\partial x} = 0$$

This all implies that $\eta \ll h$. We can get a wave equation for this with for flat bottom

$$\eta_{tt} - gh\eta_{xx} = 0 \tag{21.1}$$

21.3 Wave Equation for Simple Steady and Uniform Flow

Now, consider a case where there is a steady and spatially uniform mean flow in the x direction U. Then the total current in the x direction is $U + u$. Lets say that $U \gg u$. Doing a similar linearization procedure gives us

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} + h\frac{\partial u}{\partial x} = 0$$

Now define a new time derivative operator

$$\partial_{\tilde{t}} = \partial_t + U\partial x, \tag{21.2}$$

the equations above revert back to the linearized about the mean shallow water equations with wave equation of

$$\eta_{\tilde{t}\tilde{t}} - gh\eta_{xx} = 0 \tag{21.3}$$

and with

$$\eta_{tt} + 2U\eta_{xt} + U^2\eta_{xx} - gh\eta_{xx} \tag{21.4}$$

and with plugging in solution $\eta = \hat{\eta}\exp[\mathrm{i}(kx - \omega t)]$ we get

$$-\omega^2 + 2U\omega k + U^2k^2 + ghk^2 \tag{21.5}$$

Now if we complete the square we get

$$(\omega - Uk)^2 = ghk^2 \tag{21.6}$$

and if we define the intrinsic frequency $\omega_i = \sqrt{gh}k$ as the dispersion relationship without any mean flow, Now we replace the symbol ω bu ω_a to connote the absolute frequency, ie that seen by a fixed observer. Then we have

$$(\omega_a - Uk)^2 = \omega_i^2 \tag{21.7}$$

$$\omega_a - Uk = \omega_i \tag{21.8}$$

$$\omega_a = Uk + \omega_i. \tag{21.9}$$

This implies the absolute frequency (ω_a) is Doppler shifted by the mean currents (Uk) away from the intrinsic frequency (ω_i). This can be generalized to two dimensions with

$$\omega_a = \mathbf{U} \cdot \mathbf{k} + \omega_i$$

This kind of analysis can apply to many types of wave systems.

21.4 Slowly varying mean currents